FROM DYSON–SCHWINGER EQUATIONS TO QUANTUM ENTANGLEMENT

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ABSTRACT. We apply combinatorial Dyson–Schwinger equations and their Feynman graphon representations to study quantum entanglement in a gauge field theory Φ in terms of cut-distance regions of Feynman diagrams in the topological renormalization Hopf algebra $H_{\rm FG}^{\rm cut}(\Phi)$ and lattices of intermediate structures. Feynman diagrams in $H_{\rm FG}(\Phi)$ are applied to describe states in Φ where we build the Fisher information metric on finite dimensional linear subspaces of states in terms of homomorphism densities of Feynman graphons which are continuous functionals on the topological space $\mathcal{S}^{\Phi, M\subseteq[0,\infty)}_{\rm graphon}([0,1])$. We associate Hopf subalgebras of $H_{\rm FG}(\Phi)$ generated by quantum motions to separated regions of space-time to address some new correlations. These correlations are encoded by assigning a statistical manifold to the space of 1PI Green's functions of Φ. These correlations are applied to build lattices of Hopf subalgebras, Lie subgroups and Tannakian subcategories, derived from towers of combinatorial Dyson–Schwinger equations, which contribute to separated but correlated cut-distance topological regions. This lattice setting is applied to formulate a new tower of renormalization groups which encodes quantum entanglement of space-time separated particles under different energy scales.

CONTENTS

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1. Introduction

Recently, we applied theory of graphons to obtain some new analytic and computational tools for the study of non-perturbative solutions of quantum motions, encoded by Dyson–Schwinger equations, in gauge filed theories [25, 26, 27, 28, 29, 30, 31, 33, 34]. The space of Feynman diagrams of a gauge field theory can be topologically completed with respect to the topology of graphons [28, 29]. This topological enrichment is useful to formulate random graph representations for solutions of combinatorial Dyson–Schwinger equations [26, 28, 29, 34]. This research work presents some new applications of Feynman graphon models in designing a new algebraic approach to quantum entanglement in gauge field theories. For a gauge field theory Φ, we associate a particular class of commutative Hopf subalgebras to space-time regions. They are Hopf subalgebras of solutions of quantum motions in Φ which can be topologically completed by using the space of Feynman graphons. We introduce basic states in Φ in terms of 1PI primitive Feynman diagrams of the renormalization Hopf algebra $H_{\mathrm{FG}}(\Phi)$. We relate our new framework to Reeh–Schlieder theorem where we build non-zero correlations between separated cut-distance topological regions of Feynman diagrams, as space-time diagrams, such that each region of this type corresponds to a space-time region. Thanks to the homomorphism densities of Feynman graphons [28, 29], we introduce the Fisher information metric on finite dimensional linear subspaces of qft-states and then develop it to build a new statistical manifold associated with the space of 1PI Green's functions of Φ. Thereafter we explain the structures of lattices of Hopf subalgebras and Lie subgroups built on the cut-distance topological regions of Feynman diagrams in Φ which contribute to solutions of towers of combinatorial Dyson–Schwinger equations. These lattices encode information flow between space-time separated particles at qft-states in terms of intermediate algorithms which encode non-zero correlations between separated regions of space-time. In addition, we relate a lattice of Tannakian subcategories to quantum entanglement to build a renormalization group program. A tower of renormalization groups is built to show the compatibility of this new lattice setting under changing the energy scales in the physical theory.

1.1. **Entanglement: From quantum mechanics to quantum field theory.** Quantum mechanics associates some probability values to final states of physical systems of particles with microscopic scales in terms of the information of their initial states. The transition amplitude from the initial state to the final state is defined as a sum over all (unobserved) intermediate states. In entangled systems, the outcome measured by the first experimenter in one coordinate is interconnected with the choice made at the last moment by the second experimenter in another coordinate while no information can travel from those coordinates within a pre-stipulated period of time without moving faster than the speed of light. There exist certain observables which can not consistently be assigned values at all. The evolution of a quantum system can be encoded in terms of the Schrodinger equations between measurements at which they collapse to the eigenstates of the measured variables. In non-measurement interactions, the evolution of states can be studied in terms of a linear and unitary equation of motion while the particle pair in the Einstein–Podolsky–Rosen experiment remains in an entangled state. Therefore in a spin measurement, the pointers of the measurement tools are entangled with the particle pair in a non-separable state such that the indefiniteness of spins of particles is transmitted to the pointer's coordinate. However the notion of measurement is the key problem in this approach to quantum systems where adding a non-linear term to the Schrodinger equation can be useful to deal with this challenge. [13, 18, 19, 23]

Quantum field theory is built on the basis of quantum mechanics and special relativity where there exists an equivalence between matter and energy. Incoming particles collide and generate some outgoing particles with different nature while the momentum energy tensor is conserved. Initial and final states represent a fixed number of particles with assigned energy values. However unobserved intermediate states can involve any number of virtual particles with arbitrarily energy values which diverge when the upper limit of the energy of the virtual particles tends to infinity. If some parameters of a given physical theory (such as masses and coupling constants) are chosen as functions of the energy scale, then we can expect to generate some finite values in terms of perturbative renormalization. Strong running coupling constants in gauge field theories generate non-perturbative aspects where perturbation theory fails to compute non-perturbative parameters. Non-perturbative aspects are encoded in terms of towers of strongly coupled Dyson–Schwinger equations derived from fixed point equations of Green's functions. Solutions of these (systems of) equations are presented in terms of infinite power series of running coupling constants with higher loop order Feynman diagrams as coefficients. [9, 22, 30, 36]

Quantum entanglement in quantum field theory is one of the most important research topics in theoretical high and low energy physics [2, 3, 7, 8]. Quantum entanglement between particles in separated space-time regions has been considered in terms of tracing over the spinor or vector components and different fields as an intrinsic measurable physical property in relativistic models [35]. In this setting, a particular momentum-spin mode defined by a free single particle basis state may not be directly measurable because of renormalization. Reeh–Schlieder theorem in algebraic quantum field theory describes quantum entanglement under the notion of locality such that it involves entanglement between degrees of freedom inside an open region in the Hilbert space $\mathcal{H}$ of states and its causal complement. In this setting, bounded algebras of local operators assigned to separated space-like regions provide non-zero correlations. In other words, any open region U in Minkowski space-time is decorated by a local algebra consisting of all operators supported in U. Then it is shown that each arbitrary state in $\mathcal{H}$ of a quantum field theory can be approximated by $A_U \mid \varpi_0\rangle$ with respect to the vacuum state $\varpi_0 \in \mathcal{H}$ and some local operator A_U supported in U. The operator A_U acts on ϖ_0 to generate the state

$$\psi(x_1)...\psi(x_n) \mid \varpi_0\rangle = A_U \mid \varpi_0\rangle \tag{1}$$

such that $x_1, ..., x_n \in U$ while $\psi(x_i)$ are field operators defined in A_U. States $A_U \mid \varpi_0\rangle$ are dense in $\mathcal{H}$. The action of A_U on ϖ_0 can approximate another state in a space-like separated region V from U. In the vacuum state, two operators, which are supported in the separated space-like regions U and V, can be recognized such that they do not commute. This produces a non-zero correlation function which guarantees quantum entanglement. While the Hilbert space of quantum field theory does not factorize, Tomita–Takesaki modular operators are applied to analyze quantum entanglement. [11, 24, 37, 40]

1.2. **From Feynman diagrams to Feynman graphons.** Feynman diagrams are main tools for the combinatorial presentation of Green's functions in gauge field theories. Feynman rules associate an iterated ill-defined integral to each Feynman diagram where sub-divergences in the integral are encoded by nested loops. The superficial degree of divergence determines the type of divergence or convergence of a loop in its integral presentation.

Definition 1.1. A Feynman diagram is a space-time oriented decorated diagram Γ such that it has the following properties.

- Under charge conjugate, parity reversal and reversing the flow of time, a matter particle is exactly the same as its anti-matter particle in Γ.
- Γ has a vertex set $\Gamma^{[0]}$ which presents interactions between particles in the diagram. An interaction is allowed if it does not violate rules of the Standard Model.
- Γ has an edge set $\Gamma^{[1]}$ which presents particles. The subset $\Gamma^{[1]}_{\text{int}}$, which contains edges with beginning and ending vertices, encodes

virtual particles. The subset $\Gamma^{[1]}_{\mathrm{ext}}$, which contains edges with beginning or ending vertex, encodes elementary particles in the physical theory.
- Objects of $\Gamma^{[1]}$ are labeled by momenta information where at each vertex momentum conservation law is valid.

The BPHZ perturbative renormalization provides a recursive machinery for the removal of sub-divergences from integrals to extract some finite values. This renormalization program, built in terms of the (Bogoliubov–)Zimmermann's forest formula, is encapsulated by the coproduct

$$\Delta(\Gamma) = \Gamma \otimes \mathbb{I} + \mathbb{I} \otimes \Gamma + \sum_{\gamma \subset \Gamma} \gamma \otimes \Gamma/\gamma \tag{2}$$

such that the sum is taken over all disjoint unions of 1PI divergent proper subgraphs [4, 14, 17]. If the sum is over all proper subgraphs, then we have the core version of this coproduct which is useful for generalized physical theories such as quantum gravity. However we use the phrase "Feynman subdiagram" for those subgraphs which have at least a superficial (sub-)divergence.

The number of internal edges or the loop number, as the graduation parameters, are applied to formulate a graded connected commutative non-cocommutative Hopf algebra structure $H_{\mathrm{FG}}(\Phi)$ over the field $\mathbb{Q}$ on the set of Feynman diagrams in a gauge field theory Φ which is freely generated by 1PI Feynman diagrams. It is called Connes–Kreimer renormalization Hopf algebra of Feynman diagrams. [1, 17]

Definition 1.2. For any primitive Feynman diagram γ, define the grafting operator $B^+_\gamma : H_{\mathrm{FG}}(\Phi) \to H_{\mathrm{FG}}(\Phi)$ as a linear homogeneous operator. For each Feynman diagram Γ, B^+_γ replaces a vertex in γ with Γ in terms of the compatibility between types of vertices in γ and types of external edges in Γ.

$B^+_\gamma(\Gamma)$ is a formal linear expansion of Feynman diagrams as the result of all possible replacement choices. From the formula (2), it can be seen that

$$\Delta \circ B^+_\gamma(\Gamma) = B^+_\gamma(\Gamma) \otimes \mathbb{I} + (\mathrm{Id} \otimes B^+_\gamma) \circ \Delta(\Gamma) \ . \tag{3}$$

This recursive formula is useful to rebuild Feynman diagrams in terms of their primitive components and (quasi-)shuffle type products. [4, 14, 16]

The renormalization Hopf algebra has a universal model in terms of the space of non-planar rooted trees where the renormalization coproduct (2) is reformulated recursively in terms of the grafting operator and admissible cuts [4, 16]. The resulting combinatorial Hopf algebra H_{CK} is the graded connected finite type free commutative non-cocommutative Hopf algebra of non-planar rooted tress. Using suitable packages of decorations enables us to represent Feynman diagrams via decorated non-planar rooted trees or their linear combinations. For example, the collection of primitive 1PI Feynman diagrams or the collection of loops can be applied as labels for vertex sets of rooted trees.

Theorem 1.3.
- *Consider the category $\mathcal{C}_{\mathrm{Hopf}}$ of pairs (H, L) of commutative Hopf algebras together with Hocschild one-cocycles $L : H \to H$. A morphism $f : (H_1, L_1) \to (H_2, L_2)$ is a homomorphism of Hopf algebras such that $f \circ L_1 = L_2 \circ f$. (H_{CK}, B^+) is the initial object of $\mathcal{C}_{\mathrm{Hopf}}$.*
- *For a gauge field theory Φ, there exists an injective homomorphism of Hopf algebras from $H_{\mathrm{FG}}(\Phi)$ to $H_{\mathrm{CK}}(\Phi)$ such that trees are decorated by 1PI primitive Feynman diagrams of the physical theory.*
- *Fundamental identities between Feynman diagrams in Φ are encoded by some Hopf ideals $I(\Phi)$ of the renormalization Hopf algebra. In this case, the quotient Hopf algebras $H_{\mathrm{FG}}(\Phi)/I(\Phi)$ can be embedded into the combinatorial quotient Hopf algebras $H_{\mathrm{CK}}(\Phi)/\tilde{I}(\Phi)$.*

[14, 16, 38]

Therefore Feynman diagrams can be simplified in terms of decorated versions of rooted trees where the vertex set of each tree encodes nested loops or sub-divergences in the corresponding Feynman integral. Theory of graphons for sparse graphs ([5, 6]) allows us to associate graph functions to Feynman diagrams which leads us to formulate the notion of convergence for the sequences of Feynman diagrams with increasing loop numbers which have almost zero density.

Definition 1.4. Consider a gauge field theory Φ.

- For any Feynman diagram Γ in Φ, the adjacency matrix of its rooted tree representation t_Γ determines a bounded symmetric Lebesgue measurable function $P_\Gamma : [0,1] \times [0,1] \to \mathbb{R}$ which is called the pixel picture of Γ.
- For an invertible Lebesgue measure preserving transformation ρ on $[0,1]$, $P_\Gamma^\rho : [0,1] \times [0,1] \to \mathbb{R}$ is called a labeled Feynman graphon associated with Γ such that $P_\Gamma^\rho(x,y) := P_\Gamma(\rho(x), \rho(y))$.
- Labeled Feynman graphons Z_1 and Z_2 associated with Γ are called weakly isomorphic (i.e. $Z_1 \approx Z_2$) iff there exist Lebesgue measure preserving transformations ρ_1, ρ_2 such that $Z_1 = P_\Gamma^{\rho_1}$ and $Z_2 = P_\Gamma^{\rho_2}$ almost everywhere with respect to the Lebesgue measure.
- Define a unique equivalence class

$$[W_\Gamma]_\approx := \{Z : \ Z \approx P_\Gamma\} = \{P_\Gamma^\nu \ : \ \nu \text{ Lebesgue measure preserving transformation on } [0,1]\}. \tag{4}$$

  It is called the unlabeled Feynman graphon class associated with Γ.
- For an arbitrary ground measure space (Ω, μ_Ω) together with any μ_Ω-measure preserving transformation θ, the bounded symmetric μ_Ω-measurable function $W_\Gamma^\theta : \Omega \times \Omega \to \mathbb{R}$ is called a stretched labeled Feynman graphon associated with Γ.

Remark 1.5.
- For any Feynman digram Γ with overlapping sub-divergences, its tree representation is a linear combination $t_\Gamma = s_1 + ... + s_n$ of decorated non-planar rooted trees. In this case, any Feynman graphon W_Γ is a direct sum of Feynman graphons W_{s_i} which are defined on the intervals $I_i \subset [0,1]$ such that $I_i \cap I_j = \emptyset$ for all $i \neq j$ and $\bigcup_{i=1}^{n} I_i \subseteq [0,1]$.
- Stretched Feynman graphons are obtained in terms of the rescaling of the basic ground measure space or the canonical graphons. Feynman graphons are applied to define a new distance on the space of Feynman diagrams.

[28, 29, 33, 34]

1.3. **Dyson–Schwinger equations.** The reformulation of the BPHZ perturbative renormalization in terms of the Connes–Kreimer renormalization Hopf algebra provided several computational tools in dealing with Feynman integrals and physical parameters in renormalizable theories. In addition, the applications of this Hopf algebraic school have already been considered in non-perturbative aspects of gauge field theories where fundamental identities between Feynman diagrams and quantum motions are encapsulated in terms of Hopf ideals and Hochschild equations.

Dyson–Schwinger equations, which are generalizations of the Heisenberg equations of motion, determine quantum motions in gauge field theories. These equations, which are derived from the interaction part of the Lagrangian, contain coupled systems of integral equations generated by fixed point equations of Green's functions. Solutions of Dyson–Schwinger equations are polynomials with respect to running coupling constants. Regularization techniques replace space-time with a finite hyper-cubic lattice of points. In this setting, a continuous variable is replaced with a discrete variable with finite range, field and source functions become finite dimensional variables and the differential operator becomes a symmetric matrix. Therefore functional derivatives can be replaced with ordinary partial derivatives where the Fourier transform of the discrete version of Dyson–Schwinger equations generates a set of first-order partial differential equations [22, 36]. Under a different setting, Dyson–Schwinger equations are reformulated in terms of the renormalization coproduct and Hochschild cohomology theory [17, 39].

Definition 1.6. Consider a gauge field theory Φ. Define a complex $(\{C_n\}_{n\geq 0}, \mathbf{b})$. For each n, let C_n be the set of all linear maps from $H_{\mathrm{FG}}(\Phi)$ to $H_{\mathrm{FG}}(\Phi)^{\otimes n}$ such that C_0 is the field with characteristic zero. Define a coboundary operator $\mathbf{b}$ given by

$$\mathbf{b}T := (\mathrm{Id} \otimes T)\Delta + \sum_{i=1}^{n} (-1)^i \Delta_i T + (-1)^{n+1} T \otimes \mathbb{I} \ . \tag{5}$$

The Hochschild cohomology of the complex $(\{C_n\}_{n\geq 0}, \mathbf{b})$ encodes required information for the reconstruction of Dyson–Schwinger equations. For any primitive Feynman diagram γ, $\mathbf{b}B^+_\gamma = 0$ which means that it is a Hochschild one-cocycle. The grafting operators associated with primitive Feynman diagrams determine an important family of generators of the first rank cohomology group.

Definition 1.7. For a given family $\{B^+_{\gamma_n}\}_{n\geq 1}$ of Hochschild one-cocycles corresponding to primitive Feynman diagrams $\{\gamma_n\}_{n\geq 1}$, the combinatorial recursive equation

$$\text{DSE}: \quad X = \mathbb{I} + \sum_{n\geq 1} (\lambda g)^n \omega_n B^+_{\gamma_n}(X^{n+1}) \ , \ 0 < \lambda \leq 1 \tag{6}$$

identifies a class of Dyson–Schwinger equations underlying the running coupling constant λg. It is called a combinatorial Dyson–Schwinger equation.

For $n \geq 1$, ω_n are some constants. We can replace $(\lambda g)^n \omega_n$ with $c(g) w_n$ such that $c(g)$ is a function of the bare coupling constant generated by regularization techniques and $w_n := c(g)^{n-1}\omega_n$. Therefore the equation (6) encodes Dyson–Schwinger equations under all possible running coupling constants.

Remark 1.8.
- The unique solution $X_{\text{DSE}} = \sum_{n\geq 0} (\lambda g)^n X_n$ of an equation DSE is given by the recursive relations
$$X_n = \sum_{j=1}^{n} \omega_j B^+_{\gamma_j}\Big(\sum_{k_1+...+k_{j+1}=n-j, \ k_i \geq 0} X_{k_1} ... X_{k_{j+1}} \Big) \tag{7}$$
such that X_0 is the empty graph.
- The large Feynman diagram X_{DSE} is an object in the completion of $H_{\text{FG}}(\Phi)[[\lambda g]]$ with respect to the n-adic topology.
- Feynman diagrams $X_n \in H_{\text{FG}}(\Phi)$ are generators of a graded connected free commutative Hopf subalgebra H_{DSE} of $H_{\text{FG}}(\Phi)$. This Hopf subalgebra is classified by the Faa di Bruno Hopf algebra and the Hopf algebra of symmetric functions.
- Non-linear Dyson–Schwinger equations, derived from strong (running) coupling constants, determine non-cocommutative Hopf subalgebras while linear Dyson–Schwinger equations, derived from physical theories with the vanishing β-function, determine cocommutative Hopf subalgebras.

[10, 15, 17, 39]

Solutions of quantum motions under strong coupling constants generate infinite formal expansions of powers of coupling constants together with Feynman integrals with nested sub-divergences as coefficients. In combinatorial setting, these Feynman integrals are replaced with higher loop orders Feynman diagrams. Definition 1.7 and Remark 1.8 are applied to represent these non-perturbative expansions in terms of infinite direct sums of stretched Feynman graphon models (i.e. Theorem 2.10). The completeness of the space of stretched Feynman graphons is applied for the interpretation of the unique solution of a combinatorial Dyson–Schwinger equation as the cut-distance convergent limit of a sequence of random graph processes. As the consequence, a new non-perturbative renormalization program on the space of quantum motions are achieved. [29, 30, 31, 33, 34]

1.4. **Original achievements.** Consider a (strongly coupled) interacting gauge field theory Φ. Thanks to the cut-distance topological space of Feynman diagrams (i.e. Definitions 1.4 and 2.5 and Remark 2.6), the renormalization Hopf algebraic approach to quantum motions (i.e. Definitions 1.2 and 1.7 and Remark 1.8) and Feynman graphon representations of solutions of combinatorial Dyson–Schwinger equations (i.e. Definition 1.4 and Theorem 2.10), here we explain a new mathematical setting for the analysis of quantum entanglement in Φ in terms of separated but correlated cut-distance topological regions of Feynman diagrams which contribute to solutions of towers of combinatorial Dyson–Schwinger equations.

- Primitive Feynman diagrams in $H_{\mathrm{FG}}(\Phi)$ (determined by Formula (2)) and the grafting operators (i.e. Definition 1.2) are applied to introduce qft-states in Φ (i.e. Definitions 1.1 and 2.3). We apply homomorphism densities of Feynman graphons (i.e. Definition 2.7) to formulate the Fisher information metric on any finite dimensional subspaces of qft-states (i.e. Theorem 2.9).
- It is observed that Feynman diagrams, which contribute to solutions of quantum motions in Φ, are rich enough to encapsulate building blocks of information flow between separated regions of space-time. Feynman graphons in the metric space $\mathcal{S}^{\Phi,[0,1)}_{\mathrm{graphon},\approx}([a,b])$, which encode solutions of quantum motions, are applied to associate some new topological algebras to space-time regions. See Remark 2.4 and Theorems 2.11 and 2.12 and Corollary 2.13.
- On the one hand, we associate a statistical manifold equipped with the Fisher information metric (37) to the space of 1PI Green's functions of Φ to present correlations between separated regions in the topological renromalization Hopf algebra in terms of homomorphism densities of partial expansions of 1PI Green's functions with respect to Feynman graphons. On the other hand, towers of combinatorial Dyson–Schwinger equations in Φ are built in terms of the recursive process (39). Hopf subalgebras of solutions of these equations together with their Hopf sub-subalgebras are arranged in some lattice structures. These particular lattices, which are built on the cut-distance topological space of Feynman diagrams, enable us to mathematically encode information flow between space-time separated particles at different qft-states (i.e. Theorems 3.1 and 3.4). Therefore, correlations encoded by homomorphism densities together with lattices of Hopf subalgebras generated by towers of combinatorial Dyson–Schwinger equations provide correlations between separated

regions of space-time. See Theorem 2.16, Corollaries 2.13, 3.3 and 3.6.

- Lattices of Hopf subalgebras of quantum motions are lifted onto the level of Lie (sub)groups to analyze quantum entanglement in terms of intermediate algorithms in theory of computation and Galois theory. See Corollaries 4.1 and 4.2.
- Lattices of Hopf subalgebras of quantum motions are lifted onto the level of Tannakian subcategories (i.e. Lemmas 5.1 and 5.2) to provide a new geometric interpretation of quantum entanglement in gauge field theories in terms of a new tower of renormalization groups. See Theorem 5.3 and Corollaries 5.4 and 5.5.

2. Correlations between separated regions in $H^{\mathrm{cut}}_{\mathrm{FG}}(\Phi)$

Single-particle states of a non-interacting particle are labeled by quantum numbers, namely momentum k and other physical parameters u such as charge, spin, etc. The Hilbert space of states corresponding to this particle is given by the span $\mathcal{H}_1 = \{| \ \psi\rangle = \sum_{k,u} \alpha_{k,u} \mid ku\rangle\}$ such that $\alpha_{k,u}$ are complex coefficients with respect to all possible values of k and u. The N-particle Hilbert space $\mathcal{H}_N$, as a product of N copies of $\mathcal{H}_1$, is determined in terms of the span of all possible N-particle states. While a pure state is given when the probability $p_{i_0} = 1$ for some state i_0, the state of a quantum system is not known and it is determined in terms of a linear combination of basic states $\mid n\rangle$ with the probabilities p_n such that $\sum_n p_n = 1$. This means that some expectation values can be assigned to observables. [18]

The Hilbert space of quantum field theory includes the space of no-particle states and the space of all possible N-particle states for any integers $N \geq 1$ where N tends to infinity. Therefore this infinite dimensional separable Hilbert space is a direct sum of subspaces associated with vacuum, one-particle, two-particles, etc. The interaction terms in Hamiltonian generate additional particles in 0-particle and one-particle states.

In this section, we review the structure of combinatorial Green's functions to describe states in a gauge field theory Φ in terms of the objects of the renormalization Hopf algebra. Then we apply Feynman graphon representations of solutions of combinatorial Dyson–Schwinger equations to address correlations between separated cut-distance topological regions of Feynman diagrams in the physical theory.

2.1. **Fisher information metric between states in a gauge field theory via Feynman graphons.** Creation and annihilation operators are two basic tools in quantum field theory where we concern quantum systems for which the number of particles can change. Any operator can be presented as a sum of products of creation and annihilation operators. The notation $\mid (k_1,u_1)_{n_1};...;(k_l,u_l)_{n_l}\rangle$ presents a N-particle state in which each of its single-particle state $\mid k_i u_i\rangle$ is occupied by n_i particles such that $N = \sum_{i=1}^{l} n_i$. The creation operator adds one particle to any given state while the annihilation operator removes one particle with a specific quantum number if this particle is present. Any multiple-particle state is obtained in terms of applying a combination of creation operators to the vacuum. This setting works for bosons as particles which can multiply occupy a state. The creation and annihilation operators for fermions are defined in terms of four cases generated by unoccupied and occupied states. The resulting fermionic states are antisymmetric under particle interchange. [22]

A space-time field is a trajectory $t \mapsto \phi(t, \bullet)$ in the space of Klein–Gordon fields as classical fields. A time-ordered Green's function is defined by

$$G_N(x_1, ..., x_N) = \langle 0 \mid T\phi(x_1)...\phi(x_N) \mid 0\rangle \tag{8}$$

such that 0 is the vacuum state. The canonical quantization of Klein–Gordon fields gives their corresponding quantum fields $\tilde{\phi}(t,x) = e^{itH}\tilde{\phi}(x)e^{-itH}$ with respect to the Hamiltonian H. These quantum fields are operators which can be presented in terms of linear combinations of creation and annihilation operators. Quantum fields act on the Fock space which is an infinite direct sum of (anti)symmetrized Hilbert spaces corresponding to a fixed number of particles. The quantum expectation value of a classical observable is given by $\langle \mathcal{O} \rangle = \mathfrak{n} \int \mathcal{O}(\phi)\exp\Big(i\frac{S(\phi)}{\hbar}\Big)\mathcal{D}[\phi]$ such that $S(\phi) = S_0(\phi) + S_{\text{int}}(\phi)$ is the action functional, $\mathfrak{n}$ is the normalization factor and $\mathcal{D}$ is the measure on the linear space of classical fields.

Definition 2.1. Green's functions, as integral over histories, are formulated in terms of formal expansions of integral polynomials. We have

$$G_N(x_1, ..., x_N) = \mathfrak{n} \int \exp\Big(i\frac{S(\phi)}{\hbar}\Big)\phi(x_1)...\phi(x_N)\mathcal{D}[\phi]$$

$$= \Big(\sum_{n=0}^{\infty}\frac{i^n}{n!}\int \phi(x_1)...\phi(x_N)S_{\text{int}}(\phi)^n d\mu\Big).\Big(\sum_{n=0}^{\infty}\frac{i^n}{n!}\int S_{\text{int}}(\phi)^n d\mu\Big)^{-1} = \tag{9}$$

$$\Big(\sum_{n=0}^{\infty}\frac{i^n}{n!}\int \langle 0 \mid T\tilde{\phi}_0(x_1)...\tilde{\phi}_0(x_N)\prod_j \mathcal{L}_{\text{int}}(y_j) \mid 0\rangle \prod_j dy_j\Big).\Big(\sum_{n=0}^{\infty}\frac{i^n}{n!}\int \langle 0 \mid T\prod_j \mathcal{L}_{\text{int}}(y_j) \mid 0\rangle \prod_j dy_j\Big)^{-1} \tag{10}$$

such that $\hbar = 1$, $\langle 0 \mid 0\rangle = 1$, $d\mu = \exp\Big(iS_0(\phi)\Big)\mathcal{D}[\phi]$, $\mathcal{L}_{\text{int}}$ is the interaction part of the Lagrangian density and $\tilde{\phi}_0$ is a free field. [9, 22]

Excitations of quantum fields of space-time determine elementary particles such that the ground state is considered as the vacuum. Non-zero energy of the vacuum in interacting theories can be interpreted as creation–annihilation of particle–antiparticle pairs. The vacuum state in free field theory is described as a tensor product of the Fock space vacuum states for each independent field mode. So there is no entanglement between the field modes at different momenta. However in interacting theories, the full vacuum state can be interpreted as a superposition of the Fock basis states which means that the modes of different momenta are entangled. States in interacting gauge field theories encode information about these excitations. Each state has information about the number of created particles from each type of field, momentum, spin, etc. Therefore we can describe a state in terms of a linear combination of the possibilities corresponding to 1-particle, 2-particle,..., N-particle wavefunctions such that N tends to infinity. The computation of a transition amplitude provides only a part of a quantum state.

Remark 2.2. Perturbation theory concerns intermediate virtual states which have different types than the initial and final states. These types of states have uncertain energy scale, but since they are available at only very short time, they can have the same energy as the initial and final states. Therefore, without violating the energy conservation, these types of states, generated by virtual particles, can be considered with some probabilities.

Green's functions are computed in terms of integration by parts on (10) where a large number of terms are generated. Following Definition 1.1 together with Feynman rules of the physical theory, we associate Feynman diagrams to these terms to formulate a combinatorial version of Green's functions. However the computation of the self-energy of the interaction of a particle generates increasingly number of virtual particles at some energy scales where perturbation theory fails. Feynman rules in space configuration associate some propagators to vertices and edges in Γ. These rules label the space-time points of all vertices in Γ and then integrate over them. Therefore we have integrals over all possible space-time positions of all vertices. Fourier transform is applied to interpret Feynman rules in momentum configuration where these rules associate some propagators and momenta information to all internal edges with respect to the momentum conservation at each vertex.

Definition 2.3. Consider a given gauge field theory Φ with the corresponding renormalization Hopf algebra $H_{\mathrm{FG}}(\Phi)$.

- Any 1PI primitive Feynman diagram γ identifies a basic qft-state s_γ. The number $n_\gamma := |\gamma^{[1]}|$ of edges in γ presents the number of particles which occupy the basic qft-state s_γ.
- The subset $\gamma^{[1]}_{\mathrm{int}}$ encodes the intermediate qft-states corresponding to γ.
- Let Γ be a finite connected Feynman diagram built by primitive components $\gamma_1, ..., \gamma_s$. The notation $[s_{\gamma_1}^{n_{\gamma_1}}; ...; s_{\gamma_s}^{n_{\gamma_s}}]$ presents a general qft-state in which each basic qft-state s_{γ_i} is occupied by n_{γ_i} particles.
- For any 1PI primitive Feynman diagram γ, the grafting operator B^+_γ adds a basic qft-state to any general qft-state.

Remark 2.4.
- The vacuum in Φ is a homogeneous system of virtual particles such that its associated intermediate qft-states are invariant

with respect to the transformations of the invariance group. Without violating the conservation law of energy, particles with negative energy are created and annihilated in the vacuum.

- Each general qft-state associated with a finite Feynman diagram can be occupied by a finite number of particles.
- A large Feynman diagram X_{DSE} as the solution of a combinatorial Dyson–Schwinger equation DSE under a strong running coupling constant $\lambda g \geq 1$ (given by Definition 1.7 and Remark 1.8) is built in terms of an infinite number of applying the grafting operators. Therefore the general qft-state associated with X_{DSE} can be occupied with an infinite number of particles.

Consider a gauge field theory Φ and the interval $M \subseteq [0, \infty)$ as the Lebesgue measure space.

Definition 2.5.
- Feynman diagrams Γ_1, Γ_2 are called weakly isomorphic (or weakly equivalent) iff $[W_{\Gamma_1}]_{\approx} = [W_{\Gamma_2}]_{\approx}$. In other words, $\Gamma_1 \approx \Gamma_2$ iff theres exists a symmetric bounded Lebesgue measurable graph function $W : M \times M \to \mathbb{R}$ and Lebesgue measure preserving transformations ρ_1, ρ_2 on M such that $W^{\rho_1} = W_{\Gamma_1}$ and $W^{\rho_2} = W_{\Gamma_2}$ almost everywhere with respect to the Lebesgue measure. This means that $W^{\rho_1} \in [W_{\Gamma_1}]_{\approx}$ and $W^{\rho_2} \in [W_{\Gamma_2}]_{\approx}$.
- Thanks to the cut-distance topology on the space of graphons ([12]), define a distance between Feynman diagrams. For each $\Gamma_1, \Gamma_2 \in H_{\mathrm{FG}}(\Phi)$, define

(11)
$$d_{\mathrm{cut}}(\Gamma_1, \Gamma_2) := d_{\mathrm{cut}}([W_{\Gamma_1}]_{\approx}, [W_{\Gamma_2}]_{\approx}) = \inf_{\rho,\psi} \sup_{A,B \subsetneq M} \left| \int_{A \times B} \Big(W^{\rho}_{\Gamma_1}(x,y) - W^{\psi}_{\Gamma_2}(x,y) \Big) dxdy \right|$$

such that the infimum is over all Lebesgue measure preserving transformations ρ, ψ on M and the supremum is over all Lebesgue measurable subsets A, B in the interval M.
- Set $\mathcal{S}^{\Phi,M}_{\mathrm{graphon}}([a,b])$ as the cut-distance topological space of (stretched) Feynman graphons on the ground measure space M with values in $[a,b] \subset \mathbb{R}$ corresponding to Feynman diagrams of the physical theory.

Remark 2.6.
- $\mathcal{S}^{\Phi,[0,1]}_{\mathrm{graphon}}([0,1])$ is a compact Hausdorff metrizable topological space while $\mathcal{S}^{\Phi,[0,\infty)}_{\mathrm{graphon}}([a,b])$ is a complete Hausdorff metrizable topological space. We use the notations $\mathcal{S}^{\Phi,[0,1]}_{\mathrm{graphon},\approx}([0,1])$ and $\mathcal{S}^{\Phi,[0,\infty)}_{\mathrm{graphon},\approx}([a,b])$ for the corresponding metric spaces of unlabeled (stretched) Feynman graphon classes.
- Feynman diagrams Γ_1, Γ_2 are weakly isomorphic iff $d_{\mathrm{cut}}(\Gamma_1, \Gamma_2) = 0$.
- Up to the weakly isomorphic equivalence relation, a sequence $\{\Gamma_n\}_{n\geq 1}$ of Feynman diagrams is called convergent, if the sequence $\{\frac{[W_{\Gamma_n}]_{\approx}}{\|[W_{\Gamma_n}]_{\approx}\|_{\mathrm{cut}}}\}_{n\geq 1}$ converges to a non-zero Feynman graphon with respect to the metric

(11) such that for each $n \geq 1$,

$$||[W_{\Gamma_n}]_{\approx}||_{\text{cut}} := \sup_{A,B \subsetneq M} \left| \int_{A \times B} W_{\Gamma_n}(x,y) dx dy \right| . \tag{12}$$

- The renormalization Hopf algebra can be topologically completed with respect to the metric (11). The resulting enriched space is presented by $H^{\text{cut}}_{\text{FG}}(\Phi)$.

[28, 29, 31, 34]

Definition 2.7. The homomorphism density of a connected 1PI Feynman diagram $\Gamma \in H_{\text{FG}}$ with some nested loops but without overlapping loops with respect to any Feynman graphon W_X in $\mathcal{S}^{\Phi,[0,1]}_{\text{graphon}}([0,1])$ is given by

$$t(\Gamma, W_X) = \int_{[0,1]^{|t_\Gamma|}} \prod_{e_{kl} \in E(t_\Gamma)} W_X(x_k, x_l) \prod_{e_{kl} \notin E(t_\Gamma)} \Big(1 - W_X(x_k, x_l) \Big) \prod_{k=1}^{|t_\Gamma|} dx_k \ , \tag{13}$$

such that t_Γ is a decorated non-planar rooted tree.

If Γ has overlapping sub-divergences, then its tree representation t_Γ is a linear combination of decorated non-planar rooted trees. Then we have

$$t(\Gamma, W_X) = \prod_{i=1}^{n} \int_{[0,1]^{|s_i|}} \prod_{e_{kl} \in E(s_i)} W_X(x_k, x_l) \prod_{e_{kl} \notin E(s_i)} \Big(1 - W_X(x_k, x_l) \Big) \prod_{k=1}^{|s_i|} dx_k \tag{14}$$

such that $t_\Gamma = s_1 + ... + s_n$.

Corollary 2.8. *Consider a sequence $\{W_{\Gamma_n}\}_{n \geq 1}$ and W_X, W_Y of Feynman graphons in $\mathcal{S}^{\Phi,[0,1]}_{\text{graphon},\approx}([0,1])$. Let $\Gamma \in H_{\text{FG}}(\Phi)$ be any connected 1PI Feynman diagram with some nested loops but without overlapping loops.*

- *$\{W_{\Gamma_n}\}_{n \geq 1}$ is a Cauchy sequence with respect to the cut-distance metric (11) iff the sequence $\{t(\Gamma, W_{\Gamma_n})\}_{n \geq 1}$ of homomorphism densities is convergent.*
- *$\{W_{\Gamma_n}\}_{n \geq 1}$ converges to W_X with respect to the cut-distance topology iff $\{t(\Gamma, W_{\Gamma_n})\}_{n \geq 1}$ converges to $t(\Gamma, W_X)$.*
- *W_X, W_Y are weakly isomorphic iff $t(\Gamma, W_X) = t(\Gamma, W_Y)$.*

Proof. Thanks to Definitions 1.4, 2.5 and 2.7 and Remark 1.5, it is a direct result of Counting Lemma for Graphs [6]. □

Theorem 2.9. *Consider a gauge field theory Φ with the renormalization Hopf algebra $H_{\text{FG}}(\Phi)$. There exists the Fisher information metric on any linear subspace of intermediate and general states in Φ generated by a finite number of qft-states.*

Proof. Thanks to Definition 2.3 and Remark 2.4, let $S_{\Phi,n}$ be a finite dimensional vector space of states in Φ generated by n number of qft-states corresponding to finite connected Feynman diagrams $\Gamma_1, ..., \Gamma_n$. Let for each

$1 \leq i \leq n$, Γ_i is built in terms of 1PI primitive components $\gamma_{i_1}, ..., \gamma_{i_k}$. Up to the weakly isomorphism relation on the space of Feynman diagrams (i.e. Remark 2.6), define a distribution

$$h(x; W_{\Gamma_1}, ..., W_{\Gamma_n}) := \sum_{i=1}^{n} t(\Gamma_i^{\lceil x \rceil}, \oplus_{s=1}^{n} W_{\Gamma_s^{\lceil x \rceil}}) W_{\Gamma_i} \tag{15}$$

such that

- $x \in J \subset [1, \infty)$ is a random variable chosen from a compact interval J,
- $x \mapsto \lceil x \rceil$ is the ceiling function,
- For each $1 \leq i \leq n$, $\Gamma_i^{\lceil x \rceil}$ is a Feynman subdiagram of Γ_i generated by $\gamma_{i_1}, ..., \gamma_{i_{\lceil x \rceil}}$ for $\lceil x \rceil \leq k$. For $\lceil x \rceil > k$, consider the empty graph $\mathbb{I}$ as $\gamma_{i_{\lceil x \rceil}}$,
- W_{Γ_i} is the normalized stretched Feynman graphon of Γ_i and $W_{\Gamma_s^{\lceil x \rceil}}$ is the normalized stretched Feynman graphon of the Feynman subdiagram $\Gamma_s^{\lceil x \rceil}$ of Γ_s.

The Gibbs measure of the distribution (15) is given by

$$\mathbf{P}(x; W_{\Gamma_1}, ..., W_{\Gamma_n}) = \frac{1}{Z[\mathfrak{j}]} \exp\Big(- \mathfrak{j} h(x; W_{\Gamma_1}, ..., W_{\Gamma_n}) \Big) \tag{16}$$

such that

$$Z[\mathfrak{j}] = \int \exp\Big(iS(\phi) + i \int \mathfrak{j}(x)\phi(x) d^D x \Big) \mathcal{D}[\phi] \tag{17}$$

is the partition function with respect to the source field $\mathfrak{j}$ in Φ. Up to the weakly isomorphism relation on the space of Feynman diagrams (i.e. Remark 2.6), set $(W_{\Gamma_1}, W_{\Gamma_2}, ..., W_{\Gamma_n})$ as the coordinate system on $S_{\Phi,n}$ and define the metric structure with respect to the probability distribution $\mathbf{P}$ on $S_{\Phi,n}$ given by

$$g_{st}(W_{\Gamma_1}, ..., W_{\Gamma_n}) := \int_J \frac{\partial \log \mathbf{P}(x; W_{\Gamma_1}, ..., W_{\Gamma_n})}{\partial W_{\Gamma_s}} \frac{\partial \log \mathbf{P}(x; W_{\Gamma_1}, ..., W_{\Gamma_n})}{\partial W_{\Gamma_t}} \mathbf{P}(x; W_{\Gamma_1}, ..., W_{\Gamma_n}) dx \tag{18}$$

such that $1 \leq s, t \leq n$. This is the Fisher information metric on the subspace $S_{\Phi,n}$ of states in Φ generated by qft-states $\Gamma_1, ..., \Gamma_n$. □ □

2.2. **Correlations via combinatorial Dyson–Schwinger equations.** In this part, we introduce a dictionary between our platform and Reeh–Schlieder theorem. The Hilbert space $\mathcal{H}$ of states and its open subspaces are replaced with the topological renormalization Hopf algebra $H_{\rm FG}^{\rm cut}(\Phi)$ and topological Hopf subalgebras $H_{\rm DSE}^{\rm cut}$ corresponding to quantum motions. Local operators in $B(\mathcal{H})$, which provide approximations, are replaced with the grafting operators, as Hochschild one-cocycle operators, and their compositions.

Theorem 2.10. *Consider a combinatorial Dyson–Schwinger DSE given by Definition 1.7 with the solution $X_{\rm DSE}$.*

- *For each $m \geq 1$, a stretched Feynman graphon of the partial sum $Y_m := \sum_{j=0}^{m} (\lambda g)^j X_j$ is a bounded symmetric Lebesgue measurable function $\tilde{W}_{Y_m} \in \mathcal{S}^{\Phi,[0,\infty)}_{\rm graphon}([a,b])$ defined on $\bigsqcup_{j=1}^{m} I_j \times \bigsqcup_{j=1}^{m} I_j$ such that for $l \neq k$, $I_l \cap I_k = \emptyset$ and $m(I_j) = (\lambda g)^j$. Lebesgue measure preserving transformations on $[0,\infty)$ are applied to project the partition $\{I_j\}_{j=1}^{m}$ to the partition $\{\tilde{I}_j\}_{j=1}^{m} \subseteq [0,1)$ and define the normalized direct sum*

$$W_{Y_m} = \frac{\tilde{W}_{X_1} + ... + \tilde{W}_{X_m}}{||\tilde{W}_{X_1} + ... + \tilde{W}_{X_m}||_{\rm cut}} \tag{19}$$

  of stretched Feynman graphons of the components X_j. For each $1 \leq j \leq m$, $\tilde{W}_{X_j} : \tilde{I}_j \times \tilde{I}_j \rightarrow \mathbb{R}$, as a stretched version of the canonical Feynman graphon W_{X_j}, is of weight $m(\tilde{I}_j)$. For each $m \geq 1$, $[W_{Y_m}]_{\approx} \in \mathcal{S}^{\Phi,[0,1)}_{\rm graphon,\approx}([0,1])$.
- *For each m, W_{Y_m} determines a random graph $R(Y_m)$ in $[0,1]$ with $|t_{Y_m}|$ number of vertices such that with the probability $W_{Y_m}(x_i,x_j)$, there exists an edge between x_i, x_j in $R(Y_m)$. The sequence $\{R(Y_m)\}_{m\geq 1}$ of random graphs is cut-distance convergent to the normalized stretched Feynman graphon $W_{\rm DSE} \in \mathcal{S}^{\Phi,[0,1)}_{\rm graphon}([0,1])$ when m tends to infinity.*
- *The sequence $\{Y_m\}_{m\geq 1}$ of partial sums is cut-distance convergent to $X_{\rm DSE}$.*

[28, 29, 34]

Up to the weakly isomorphic relation, $[W_{\rm DSE}]_{\approx}$ is the unique unlabeled Feynman graphon class associated with the large Feynman diagram $X_{\rm DSE}$. Consider $R(X_{\rm DSE})$ as an infinite random graph in $[0,1]$ by selecting randomly infinite countable nodes $\{x_n\}_{n\geq 1}$ from $[0,1]$ such that with the probability $W_{\rm DSE}(x_i,x_j)$, there exists an edge between x_i, x_j. Non-perturbative solutions of the equation DSE under strong running coupling constants can be represented by $R(X_{\rm DSE})$. These infinite random graphs, homomorphism densities of Feynman graphons and the topological enrichment of the renormalization Hopf algebra provided some new combinatorial and analytic tools for the study of quantum motions where $H^{\rm cut}_{\rm FG}(\Phi)$ is rich enough to encode solutions of quantum motions in a strongly coupled gauge field theory Φ. [25, 27, 28, 29, 30, 31]

Consider a combinatorial Dyson–Schwinger equation DSE generated by the family $\{\gamma_n\}_{n\geq 1}$ of (1PI) primitive Feynman diagrams with the solution $X_{\rm DSE}$ and the sequence $\{Y_m\}_{m\geq 1}$ of its partial sums in a gauge filed theory Φ.

Theorem 2.11. *Following Definition 2.3, each general qft-state corresponding to DSE can be approximated by grafting operators.*

Proof. Consider $H^{\rm cut}_{\rm DSE}$ as the topological algebra for cut-distance regions of Feynman diagrams which contribute to the sequence $\{Y_m\}_{m\geq 1}$ of partial sums. For each (1PI) primitive Feynman diagram $\gamma_{i_l} \in H_{\rm DSE}$, the grafting operator $B^+_{\gamma_{i_l}} : H^{\rm cut}_{\rm DSE} \to H^{\rm cut}_{\rm DSE}$ is a bounded continuous operator such that its action on the empty graph $\mathbb{I}$ generates general qft-states of the form $B^+_{\gamma_{i_l}} B^+_{\gamma_{i_{l-1}}} ... B^+_{\gamma_{i_1}}(\mathbb{I})$. Each Feynman diagram Γ can be presented by $\Gamma = B^+_{\gamma,G}(\Gamma')$ for some primitive Feynman diagram γ where Γ' is a forest of nested sub-divergences of Γ such that after shrinking all of them to a point in Γ, γ remains [17]. The symbol G encodes information about where to insert these sub-divergences in γ to rebuild Γ. Therefore all general qft-states can be presented as linear combinations of the multiple times applications of the grafting operators. □ □

Theorem 2.12. *Each equation DSE determines non-zero correlations between separated space-like regions.*

Proof. Relation 3 shows the non-commutativity of the grafting operators with respect to each other. In other words, $B^+_{\gamma_{i_l}} B^+_{\gamma_{i_{l-1}}} ... B^+_{\gamma_{i_1}}(\mathbb{I}) \neq B^+_{\gamma_{i_1}} B^+_{\gamma_{i_2}} ... B^+_{\gamma_{i_l}}(\mathbb{I})$. For two different finite families $\{\gamma_{i_1}, ..., \gamma_{i_l}\}$ and $\{\gamma_{j_1}, ..., \gamma_{j_k}\}$ of primitive (1PI) Feynman diagrams from $\{\gamma_n\}_{n\geq 1}$, consider operators

$$B^+_{\gamma_{i_l}} B^+_{\gamma_{i_2}} ... B^+_{\gamma_{i_1}}(\mathbb{I}) \ , \ B^+_{\gamma_{j_k}} B^+_{\gamma_{j_2}} ... B^+_{\gamma_{j_1}}(\mathbb{I}) \ . \tag{20}$$

Definition 2.5 and Remark 1.8 show that

$$d_{\rm cut}\left(B^+_{\gamma_{i_l}} B^+_{\gamma_{i_2}} ... B^+_{\gamma_{i_1}}(\mathbb{I}), B^+_{\gamma_{j_k}} B^+_{\gamma_{j_2}} ... B^+_{\gamma_{j_1}}(\mathbb{I}) \right) \neq 0 \ , \ d_{\rm cut}(X_i, X_j) \neq 0 \ , \ \forall i \neq j \ . \tag{21}$$

Thanks to Definition 2.3, the operators (20) are supported in two separated space-like regions $U_{i_1 \to i_l}$ and $U_{j_1 \to j_k}$ at different qft-states.

There exist some orders N_l, N_k such that

$$Y_s \cap B^+_{\gamma_{i_l}} B^+_{\gamma_{i_2}} ... B^+_{\gamma_{i_1}}(\mathbb{I}) \neq \emptyset \ , \ s \geq N_l \ ,$$

$$Y_n \cap B^+_{\gamma_{j_k}} B^+_{\gamma_{j_2}} ... B^+_{\gamma_{j_1}}(\mathbb{I}) \neq \emptyset \ , \ n \geq N_k \ . \tag{22}$$

In addition, thanks to Theorem 2.10, for each $\epsilon > 0$, there exists an order N_ϵ such that for each $t \geq N_\epsilon$,

$$d_{\rm cut}(Y_t, X_{\rm DSE}) < \epsilon \ . \tag{23}$$

Set $N_{l,k,\epsilon} := {\rm Max}\{N_l, N_k, N_\epsilon\}$. For each $m \geq N_{l,k,\epsilon}$, the relations (22) and (23) are valid. This means that the sequence $\{Y_m\}_{m\geq N_{l,k,\epsilon}}$ of partial sums of $X_{\rm DSE}$ provide non-zero correlations between $U_{i_1 \to i_l}$ and $U_{j_1 \to j_k}$. □ □

Corollary 2.13. *There exist non-zero correlations between separated cut-distance regions in $H^{\rm cut}_{\rm FG}(\Phi)$.*

Proof. Let U, V be separated regions in $H^{\text{cut}}_{\text{FG}}(\Phi)$. Let $\text{Prim}(H_{\text{FG}}(\Phi))$ be the collection of primitive Feynman diagrams in Φ, and

$$U \cap \text{Prim}(H_{\text{FG}}(\Phi)) = \{\gamma_{s_U}\}_s \ , \ V \cap \text{Prim}(H_{\text{FG}}(\Phi)) = \{\gamma_{t_V}\}_t \ . \tag{24}$$

Thanks to Definition 1.7, build combinatorial Dyson–Schwinger equations DSE_{U} defined in terms of the family $\{B^+_{\gamma_{s_U}}\}_s$ and DSE_{V} defined in terms of the family $\{B^+_{\gamma_{t_V}}\}_t$. Consider $X_{\text{DSE}_{\text{U}}}$ and $X_{\text{DSE}_{\text{V}}}$ as the solutions of these equations. Thanks to Definition 2.5 and Theorem 2.10, define a distance function between these equations given by

$$d_{\text{cut}}(\text{DSE}_{\text{U}}, \text{DSE}_{\text{V}}) = \lim_{m\to\infty} d_{\text{cut}}(Y^{\text{U}}_m, Y^{\text{V}}_m) > 0 \ , \tag{25}$$

such that $Y^{\text{U}}_m, Y^{\text{V}}_m$ are partial sums of $X_{\text{DSE}_{\text{U}}}$ and $X_{\text{DSE}_{\text{V}}}$.

Now build a new combinatorial Dyson–Schwinger equation DSE_{UV} generated by the collection $\{\gamma_{s_U}\}_s \cup \{\gamma_{t_V}\}_t$ of primitive Feynman diagrams with the solution $X_{\text{DSE}_{\text{UV}}}$. Thanks to the metric (25), for any cut-distance neighborhood A_{UV} of $X_{\text{DSE}_{\text{UV}}}$, there exist orders N_U, N_V such that for any $m \geq N_{UV} := \text{Max}\ \{N_U, N_V\}$,

$$\{Y^{\text{U}}_m\}_{m\geq N_{UV}} \cap A_{\text{UV}} \neq \emptyset \ , \ \{Y^{\text{V}}_m\}_{m\geq N_{UV}} \cap A_{\text{UV}} \neq \emptyset \ . \tag{26}$$

Therefore different graphs $X_{\text{DSE}_{\text{U}}}$ and $X_{\text{DSE}_{\text{V}}}$ which belong to separated regions in $H^{\text{cut}}_{\text{FG}}(\Phi)$ at different qf-states are correlated via equations of type DSE_{UV}. □ □

2.3. **A statistical manifold associated with the space of 1PI Green's functions.** Homomorphism densities of Feynman graphons in $\mathcal{S}^{\Phi,[0,1]}_{\text{graphon}}([0,1])$ are considered to obtain some geometric tools for the analytic study of quantum motions and their solution spaces in a gauge field theory Φ [28, 29]. Thanks to a generalization of Stone–Weierstrass theorem, under the topology of uniform convergence, the linear span of homomorphism densities is dense in the space $\mathcal{C}\left(\mathcal{S}^{\Phi,[0,1]}_{\text{graphon},\approx}([0,1]), d_{\text{cut}}\right)$ of continuous functionals on $\mathcal{S}^{\Phi,[0,1]}_{\text{graphon},\approx}([0,1])$. In this part, thanks to Theorem 2.10, we apply homomorphism densities of Feynman graphons to determine the Fisher information metric on the space of 1PI Green's functions in Φ.

1PI Green's functions $\Gamma^{e_i}, \Gamma^{v_j}$ in Φ are characterized in terms of formal expansions of Feynman diagrams with residues $e_1, ..., e_m$ as all possible types of fermions and bosons, and $v_1, ..., v_n$ as all possible types of interactions in the physical theory. Under a combinatorial setting, we have

$$\Gamma^r(\lambda) = \mathbb{I} \pm \sum_{k=1}^{\infty} (\lambda g)^k \sum_{\Gamma, \text{res}(\Gamma)=r, |\Gamma|=k, |\Gamma|_{\text{aug}}=1} \frac{1}{\text{Sym}(\Gamma)} B^+_\Gamma(X_\Gamma) \ , \ X_\Gamma = \prod_{e\in\Gamma^{[1]}_{\text{int}}} \prod_{v\in\Gamma^{[0]}} \Gamma^v/\Gamma^e \ , \tag{27}$$

such that $r \in \mathcal{R}_\Phi := \{e_1, ..., e_m, v_1, ..., v_n\}$ [15, 38]. Fixed point equations of these 1PI Green's functions are combinatorial Dyson–Schwinger equations.

Up to the weakly isomorphic relation on the space of Feynman diagrams (given by Definition 2.5), the linear span of

$$\Big\{ \mathrm{DSE}_{e_i}(\lambda g) \ , \ \mathrm{DSE}_{v_j}(\lambda g) \ , \ e_i, v_j \in \mathcal{R}_\Phi \ , \ 0 < \lambda \leq 1 \Big\} \tag{28}$$

encodes quantum motions in the physical theory under different running coupling constants.

Definition 2.14. Consider combinatorial Dyson–Schwinger equations $\mathrm{DSE}_1, \mathrm{DSE}_2$ with the solutions $X_{\mathrm{DSE}_1}, X_{\mathrm{DSE}_2}$ and corresponding sequences of partial sums $\{Y_m^{(1)}\}_{m\geq 1}$ and $\{Y_m^{(2)}\}_{m\geq 1}$. For any $m \geq 1$ and $i, j \in \{1, 2\}$, the homomorphism density of $Y_m^{(i)}$ with respect to the normalized stretched Feynman graphon $W_{\mathrm{DSE}_j} \in \mathcal{S}^{\Phi,[0,1)}_{\text{graphon}}([0,1])$ is given by

(29)

$$t(Y_m^{(i)}, W_{\mathrm{DSE}_j}) = \int_{[0,1)^{|t_{Y_m^{(i)}}|}} \prod_{e_{kl} \in E(t_{Y_m^{(i)}})} W_{\mathrm{DSE}_j}(x_k, x_l) \prod_{e_{kl} \notin E(t_{Y_m^{(i)}})} \Big(1 - W_{\mathrm{DSE}_j}(x_k, x_l)\Big) \prod_{k=1}^{|t_{Y_m^{(i)}}|} dx_k \ ,$$

such that $t_{Y_m^{(i)}}$ is the forest representation of $Y_m^{(i)}$ in terms of decorated non-planar rooted trees.

Remark 2.15.
- Thanks to Proposition 2 in [29], the homomorphism density $t(X_{\mathrm{DSE}_i}, W_{\mathrm{DSE}_j})$ is defined in terms of the inverse limit

$$t(X_{\mathrm{DSE}_i}, W_{\mathrm{DSE}_j}) = \lim_{\leftarrow m} t(Y_m^{(i)}, W_{\mathrm{DSE}_j}) \tag{30}$$

  of the homomorphism densities of the partial sums $Y_m^{(i)}$.
- Equations DSE_1 and DSE_2 are weakly isomorphic (i.e. $d_{\mathrm{cut}}(X_{\mathrm{DSE}_1}, X_{\mathrm{DSE}_2}) = 0$) iff for any connected 1PI Feynman diagram $\Gamma \in H_{\mathrm{FG}}(\Phi)$ with some nested loops but without overlapping loops,

$$t(\Gamma, W_{\mathrm{DSE}_1}) = t(\Gamma, W_{\mathrm{DSE}_2}) \ . \tag{31}$$

Theorem 2.16. *Homomorphism densities of Feynman graphons associate a statistical manifold to the gauge field theory* Φ.

Proof. It is possible to replace combinatorial Dyson–Schwinger equations with their corresponding 1PI Green's functions. For any $r \in \mathcal{R}_\Phi$ and Feynman diagram Γ, the homomorphism density of Γ with respect to $W_{\Gamma^r(\lambda)} \in \mathcal{S}^{\Phi,[0,1)}_{\text{graphon}}([0,1])$ is given by

(32)

$$t(\Gamma, W_{\Gamma^r(\lambda)}) = \int_{[0,1)^{|t_\Gamma|}} \prod_{e_{kl} \in E(t_\Gamma)} W_{\Gamma^r(\lambda)}(x_k, x_l) \prod_{e_{kl} \notin E(t_\Gamma)} \Big(1 - W_{\Gamma^r(\lambda)}(x_k, x_l)\Big) \prod_{k=1}^{|t_\Gamma|} dx_k \ ,$$

such that t_Γ is the rooted tree representation of Γ. $t(\Gamma, W_{\Gamma^r(\lambda)})$ determines the probability of constructing a random graph $R(\Gamma)$ associated with Γ via

$W_{\Gamma^r(\lambda)}$. In other words, $t(-, W_{\Gamma^r(\lambda)})$ can be applied to define a probability distribution on the space of Feynman diagrams in Φ.

Consider 1PI Green's functions in Φ (given by Formula (27)) under the bare coupling constant g to define a statistical manifold with coordinates

$$[W_{\Gamma^r}] := (W_{\Gamma^{e_1}(1)}, ..., W_{\Gamma^{e_m}(1)}, W_{\Gamma^{v_1}(1)}, ..., W_{\Gamma^{v_n}(1)}) \ , \ r \in \mathcal{R}_\Phi \tag{33}$$

up to the weakly isomorphic relation on the space of Feynman diagrams and combinatorial Dyson–Schwinger equations (i.e. Remarks 2.6 and 2.15). Thanks to the homomorphism density (32), define a distribution $p((\lambda, x), [W_{\Gamma^r}])$ in terms of two random variables on the coordinates $[W_{\Gamma^r}]$. The one random variable is applied to regularize the bare coupling constant and the other random variable extracts finite partial sums of 1PI Green's functions. This function is given by

$$p((\lambda, x), [W_{\Gamma^r}]) := \sum_{r \in \mathcal{R}_\Phi} t(\Gamma^{r,\lceil x \rceil}(\lambda), \oplus_{s \in \mathcal{R}_\Phi} W_{\Gamma^{s,\lceil x \rceil}(\lambda)}) W_{\Gamma^r(1)} \ . \tag{34}$$

- $(\lambda, x) \in I \times J \subset [0, \infty) \times [1, \infty)$ is a random pair such that I, J are compact intervals.
- $x \mapsto \lceil x \rceil$ is the ceiling function.
- For each $r \in \mathcal{R}_\Phi$ and $\lceil x \rceil$ with $x \in J$, $\Gamma^{r,\lceil x \rceil}(\lambda)$ is a partial expansion of 1PI Green's function $\Gamma^r(\lambda)$ which contains Feynman diagrams with the loop numbers at most $\lceil x \rceil$. It is given by

$$\Gamma^{r,\lceil x \rceil}(\lambda) = \mathbb{I} \pm \sum_{k=1}^{\lceil x \rceil} (\lambda g)^k \sum_{\Gamma, \mathrm{res}(\Gamma) = r, |\Gamma| = k, |\Gamma|_{\mathrm{aug}} = 1} \frac{1}{\mathrm{Sym}(\Gamma)} B^+_\Gamma(X_\Gamma) \ . \tag{35}$$

- $\oplus_{s \in \mathcal{R}_\Phi} W_{\Gamma^{s,\lceil x \rceil}(\lambda)}$ is a normalized stretched Feynman graphon in $\mathcal{S}^{\Phi,[0,1)}_{\text{graphon}}([0,1])$ which is a finite direct sum of normalized stretched Feynman graphons $W_{\Gamma^{s,\lceil x \rceil}(\lambda)}$ associated with partial expansions $\Gamma^{s,\lceil x \rceil}(\lambda)$.

The Gibbs measure of the distribution (34) is given by

$$\mathbf{P}((\lambda, x), [W_{\Gamma^r}]) = \frac{1}{Z[\mathrm{j}]} \exp\Big(- \mathrm{j} p((\lambda, x), [W_{\Gamma^r}]) \Big) \tag{36}$$

such that $Z[\mathrm{j}]$ is the partition function with respect to the source field j in Φ. Up to the weakly isomorphic relation on the space of Feynman diagrams and combinatorial Dyson–Schwinger equations (i.e. Remarks 2.6 and 2.15), the metric structure with respect to the probability distribution $\mathbf{P}$ is given by

(37)

$$g_{st}([W_{\Gamma^r}]) := \int_{I \times J} \frac{\partial \log \mathbf{P}((\lambda, x), [W_{\Gamma^r}])}{\partial W_{\Gamma^s(1)}} \frac{\partial \log \mathbf{P}((\lambda, x), [W_{\Gamma^r}])}{\partial W_{\Gamma^t(1)}} \mathbf{P}((\lambda, x), [W_{\Gamma^r}]) \, d\lambda dx \ .$$

□ □

Remark 2.17.
- The metric (37) is the Fisher information metric on the space of 1PI Green's functions of Φ.

- Thanks to Proposition 2 in [29], the homomorphism density $t(\Gamma^r(\lambda), \oplus_{s \in \mathcal{R}_\Phi} W_{\Gamma^s(\lambda)})$, for each $r \in \mathcal{R}_\Phi$, is defined in terms of the inverse limit

$$t(\Gamma^r(\lambda), \oplus_{s \in \mathcal{R}_\Phi} W_{\Gamma^s(\lambda)}) = \lim_{\leftarrow m} t(\Gamma^{r,m}(\lambda), \oplus_{s \in \mathcal{R}_\Phi} W_{\Gamma^s(\lambda)}) \tag{38}$$

of the homomorphism densities of the partial expansions $\Gamma^{r,m}(\lambda)$.

3. A lattice model for quantum entanglement

According to Definition 2.3, a finite connected Feynman diagram Γ can be interpreted as a general qft-state built by a composition of a finite number of basic qft-states associated with 1PI primitive components of Γ. These basic qft-states are entangled because of their roles in the structure of Γ. In addition, $B^+_\gamma(\Gamma)$ generates new Feynman diagrams which are entangled to Γ. Therefore, thanks to the combinatorial reformulation of quantum motions (i.e. Definition 1.7), each combinatorial Dyson–Schwinger equation DSE determines an infinite number of entangled general qft-states via primitive Feynman diagrams $\gamma \in \{\gamma_n\}_{n \geq 1}$. They are qft-states occupied by X_{DSE} or its partial sums.

Theorem 3.1. *Quantum entanglement between a particle p at a basic qft-state γ^p and virtual particles at intermediate qft-states originated from γ^p in an interacting gauge field theory Φ can be encoded by a lattice of Hopf sub-subalgebras derived from solutions of quantum motions at qft-states originated from γ^p.*

Proof. Thanks to Definition 2.3, Theorems 2.11, 2.12 and Corollary 2.13, consider a combinatorial Dyson–Schwinger equation DSE_p defined by a family $\{\gamma_n\}_{n \geq 1}$ of 1PI primitive Feynman diagrams such that $\gamma^p \in \{\gamma_n\}_{n \geq 1}$ in terms of Definition 1.7. The grafting operator $B^+_{\gamma^p}$ generates some general qft-states occupied by some virtual particles which interact with p.

We build a tower of combinatorial Dyson–Schwinger equations originated from the initial equation DSE_p. For each $j \geq 1$, define a new collection $\{\Gamma^{(j)}_n\}_{n \geq 1}$ of graphs given by

$$\Gamma^{(j)}_n := \Gamma^{(j-1)}_1 + ... + \Gamma^{(j-1)}_n \tag{39}$$

such that $\Gamma^{(0)}_n = \gamma_n$ for each $n \geq 1$. Thanks to the Lie algebra of primitive graphs with respect to the renormalization coproduct, the linear combination of a finite number of primitive graphs is primitive.

- For each $j \geq 1$ and $n \geq 1$, $B^+_{\Gamma^{(j)}_n}$ is Hochschild one-cocycle.
- The operator $B^+_{\Gamma^{(j)}_n}$ inserts each Γ into $\Gamma^{(j-1)}_1 + ... + \Gamma^{(j-1)}_n$ with respect to types of external edges in Γ and types of vertices in $\Gamma^{(j-1)}_1 + ... + \Gamma^{(j-1)}_n$.
- For each j, $B^+_{\Gamma^{(j)}_n}(\Gamma)$ contains $B^+_{\Gamma^{(j-1)}_n}(\Gamma)$ as a subgraph.

For each $j \geq 1$, define a new combinatorial Dyson–Schwinger equation $\mathrm{DSE}_p^{(j)} :=< \{B^+_{\Gamma_n^{(j)}}\}_{n\geq 1} >$ with the corresponding Hopf subalgebra $H_{\mathrm{DSE}_p^{(j)}}$.

- For each $j \geq 1$, there exists an injective Hopf algebra homomorphism from $H_{\mathrm{DSE}_p^{(j)}}$ to $H_{\mathrm{DSE}_p^{(j+1)}}$.
- There exists the following increasing chain of Hopf subalgebras

$$H_{\mathrm{DSE}_p} \leq H_{\mathrm{DSE}_p^{(1)}} \leq H_{\mathrm{DSE}_p^{(2)}} \leq ... \leq H_{\mathrm{DSE}_p^{(j)}} \leq ... \ . \tag{40}$$

- For each $n \geq 1$, define $H(X_1^j, ..., X_n^j)$ as the commutative graded Hopf sub-subalgebra of $H_{\mathrm{DSE}_p^{(j)}}$ free generated algebraically by n elements $X_1^j, ..., X_n^j$ of the components of the solution of the equation $\mathrm{DSE}_p^{(j)}$.

Consider $\mathcal{C}_p$ as the collection of all Hopf subalgebras $H_{\mathrm{DSE}_p^{(j)}}$ and their Hopf sub-subalgebras $H(X_1^j, ..., X_n^j)$ generated by the above tower of combinatorial Dyson–Schwinger equations. Define a binary relation $\preccurlyeq$ on $\mathcal{C}_p$ given by

$$H_1 \preccurlyeq H_2 \ \Leftrightarrow \exists \ H_{i_1}, ..., H_{i_r} \in \mathcal{C}_p \tag{41}$$

such that there exist injective Hopf algebra homomorphisms

$$H_1 \to H_{i_1} \to H_{i_2} \to ... \to H_{i_r} \to H_2 \tag{42}$$

which connect H_1 to H_2. The relation $\preccurlyeq$ is reflexive, antisymmetric and transitive. So $(\mathcal{C}_p, \preccurlyeq)$, as a totally ordered set, is a lattice presented by

(43)

$$\begin{tikzcd}
H_{\mathrm{DSE}_p} \arrow[r] & H_{\mathrm{DSE}_p^{(1)}} \arrow[r] & H_{\mathrm{DSE}_p^{(2)}} \arrow[r] & ... \arrow[r] & H_{\mathrm{DSE}_p^{(j)}} \arrow[r] & H_{\mathrm{DSE}_p^{(j+1)}} \arrow[r] & ... \\
... \arrow[u] \arrow[r] & ... \arrow[u] \arrow[r] & ... \arrow[u] \arrow[r] & ... \arrow[u] \arrow[r] & ... \arrow[u] \arrow[r] & ... \arrow[u] \arrow[r] & ... \arrow[u] \\
H(X_1, ..., X_n) \arrow[u] \arrow[r] & H(X_1^1, ..., X_n^1) \arrow[u] \arrow[r] & H(X_1^2, ..., X_n^2) \arrow[u] \arrow[r] & ... \arrow[u] \arrow[r] & H(X_1^j, ..., X_n^j) \arrow[u] \arrow[r] & H(X_1^{j+1}, ..., X_n^{j+1}) \arrow[u] \arrow[r] & ... \arrow[u] \\
... \arrow[u] \arrow[r] & ... \arrow[u] \arrow[r] & ... \arrow[u] \arrow[r] & ... \arrow[u] \arrow[r] & ... \arrow[u] \arrow[r] & ... \arrow[u] \arrow[r] & ... \arrow[u] \\
H(X_1, X_2) \arrow[u] \arrow[r] & H(X_1^1, X_2^1) \arrow[u] \arrow[r] & H(X_1^2, X_2^2) \arrow[u] \arrow[r] & ... \arrow[u] \arrow[r] & H(X_1^j, X_2^j) \arrow[u] \arrow[r] & H(X_1^{j+1}, X_2^{j+1}) \arrow[u] \arrow[r] & ... \arrow[u] \\
H(X_1) \arrow[u] \arrow[r] & H(X_1^1) \arrow[u] \arrow[r] & H(X_1^2) \arrow[u] \arrow[r] & ... \arrow[u] \arrow[r] & H(X_1^j) \arrow[u] \arrow[r] & H(X_1^{j+1}) \arrow[u] \arrow[r] & ... \arrow[u]
\end{tikzcd}$$

, such that $H(X_1)$ is the greatest lower bound of this lattice. For any pair $\{H_1, H_2\}$ in $(\mathcal{C}_p, \preccurlyeq)$, if $H_1 \preccurlyeq H_2$ then define $H_1 \wedge H_2 := H_1$ and $H_1 \vee H_2 := H_2$.

The lattice (43) encodes a collection of intermediate qft-states generated by the equations $\mathrm{DSE}_p^{(j)}$ with respect to the initial basic qft-state γ^p. These intermediate qft-states are occupied by virtual particles in Feynman diagrams which contribute to solutions of $\mathrm{DSE}_p^{(j)}$ and their partial sums. The grafting operators in the structures of these solutions show that these intermediate qft-states are entangled. Therefore, thanks to Theorems 2.11 and 2.12, the tower $\{\mathrm{DSE}_p^{(j)}\}_{j\geq 1}$ together with its corresponding lattice $(\mathcal{C}_p, \preccurlyeq)$ presents an information transition package between an infinite number of virtual particles at intermediate qft-states which are entangled with the particle p at the initial basic qft-state γ^p. □ □

Remark 3.2. Following notations of Theorem 3.1 and its Proof, suppose DSE'_p is another combinatorial Dyson–Schwinger equation defined by a family $\{\gamma'_n\}_{n\geq 1}$ of 1PI primitive Feynman diagrams such that $\gamma^p \in \{\gamma'_n\}_{n\geq 1}$. Consider $\{\mathrm{DSE}_p^{',(j)}\}_{j\geq 1}$ and $(\mathcal{C}'_p, \preccurlyeq)$ as the tower and lattice built by the initial equation DSE'_p via the procedure given in Proof of Theorem 3.1.

- If equations DSE_p and DSE'_p are weakly isomorphic, i.e. $d_{\mathrm{cut}}(X_{\mathrm{DSE}_p}, X_{\mathrm{DSE}'_p}) = 0$, then the lattices $(\mathcal{C}_p, \preccurlyeq)$ and $(\mathcal{C}'_p, \preccurlyeq)$ are isomorphic.
- If $d_{\mathrm{cut}}(X_{\mathrm{DSE}_p}, X_{\mathrm{DSE}'_p}) \neq 0$, then $\{\mathrm{DSE}_p^{(j)}\}_{j\geq 1}$ and $\{\mathrm{DSE}_p^{',(j)}\}_{j\geq 1}$ determine different information transition packages.

Corollary 3.3. *Following notations of Theorem 3.1, set R^p as the collection of Feynman diagrams in Φ which contribute to solutions of all combinatorial Dyson–Schwinger equations of the type DSE_p with the corresponding tower $\{\mathrm{DSE}_p^{(j)}\}_{j\geq 1}$. Its completion with respect to the cut-distance topology is presented by $\overline{R^p}$.*

- *Thanks to Definition 2.3, Theorems 2.11 and 2.12 and Corollary 2.13, the lattice $(\mathcal{C}_p, \preccurlyeq)$ encodes quantum entanglement between all (virtual) particles which contribute to the region $\overline{R^p}$ at states generated by the basic qft-state γ^p.*
- *Thanks to Theorem 2.12 and Corollary 2.13, the cut-distance topological region $\overline{R^p}$ determines a space-time region which contains the particle p and all virtual particles which interact with p at intermediate qft-states.*

Theorem 3.4. *Quantum entanglement between space-time separated particles p at a basic qft-state γ^p and q at a basic qft-state γ^q in a (strongly coupled) interacting gauge field theory Φ can be encoded by a lattice of Hopf sub-subalgebras derived from solutions of quantum motions at qft-states originated from γ^p, γ^q and the vacuum state.*

Proof. Thanks to Definition 2.3, Theorems 2.11, 2.12 and 3.1 and Corollaries 2.13, 3.3, consider combinatorial Dyson–Schwinger equations DSE_p defined by a family $\{\gamma_n\}_{n\geq 1}$ of 1PI Feynman diagrams such that $\gamma^p \in \{\gamma_n\}_{n\geq 1}$ and DSE_q defined by a family $\{\gamma'_n\}_{n\geq 1}$ of 1PI Feynman diagrams such that $\gamma^q \in \{\gamma'_n\}_{n\geq 1}$. The grafting operators $B^+_{\gamma^p}$ and $B^+_{\gamma^q}$ generate some general qft-states occupied by some virtual particles which interact with p and q.

We determine topological regions such as $R^{c_{pq}}$ in $H^{\mathrm{cut}}_{\mathrm{FG}}(\Phi)$ such that

$$\overline{R^p} \cap \overline{R^{c_{pq}}} \neq \emptyset \ , \ \overline{R^q} \cap \overline{R^{c_{pq}}} \neq \emptyset \ . \tag{44}$$

Thanks to the metric (11), define a distance between topological regions $\overline{R^p}$ and $\overline{R^q}$ in $H^{\rm cut}_{\rm FG}(\Phi)$ given by

$$d(\overline{R^p},\overline{R^q}) := \inf\{d_{\rm cut}(X,Y): \quad X \in \overline{R^p},\ Y \in \overline{R^q}\} \ . \tag{45}$$

Thanks to Theorem 2.12 and Corollary 2.13, for $d(\overline{R^p},\overline{R^q}) > 0$, there exist some j_1, j_2 with the corresponding combinatorial Dyson–Schwinger equations ${\rm DSE}_p^{(j_1)}$ and ${\rm DSE}_q^{(j_2)}$ such that up to the weakly isomorphic relation and cut-distance topology,

$$X_{{\rm DSE}_p^{(j_1)}} = \lim_{n\rightarrow\infty} \sum_{k=0}^{n} X_k^{(j_1)}, \quad X_{{\rm DSE}_q^{(j_2)}} = \lim_{n\rightarrow\infty} \sum_{k=0}^{n} X_k^{(j_2)} \ , \tag{46}$$

$$d(\overline{R^p},\overline{R^q}) = d_{\rm cut}(X_{{\rm DSE}_p^{(j_1)}}, X_{{\rm DSE}_q^{(j_2)}}) > 0 \ . \tag{47}$$

For each $\epsilon > 0$, consider Hochschild one-cocycles of the type $B^+_{\gamma_n^\epsilon,p}$, $n \geq 1$ with the following properties.

- Each γ_n^ϵ is a finite primitive (1PI) Feynman diagram such that

$$\forall n \geq 1, \quad \gamma_n^\epsilon \notin R^p, \quad \gamma_n^\epsilon \notin R^q, \quad \gamma_n^\epsilon \in H_{\rm FG}(\Phi) \ . \tag{48}$$

- ${\rm DSE}_p^\epsilon$ is a combinatorial Dyson–Schwinger equation defined by the family $\{B^+_{\gamma_n^\epsilon,p}\}_{n\geq 1}$ and with the unique solution $X_\epsilon^p = \sum_{n\geq 0} X_n^{(\epsilon)p}$ for $\lambda g = 1$. There exists $N_\epsilon \in \mathbb{N}$ such that for each $n \geq N_\epsilon$, we have $d_{\rm cut}(X_n^{(j_1)}, X_n^{(\epsilon)p}) < \epsilon$. The triangle inequality of the cut distance metric leads us to

$$d_{\rm cut}(X_{{\rm DSE}_p^{(j_1)}}, X_\epsilon^p) < \epsilon \ . \tag{49}$$

For each $\epsilon > 0$, consider Hochschild one-cocycles of the type $B^+_{\eta_n^\epsilon,q}$, $n \geq 1$ with the following properties.

- Each η_n^ϵ is a finite primitive (1PI) Feynman diagram such that

$$\forall n \geq 1, \quad \eta_n^\epsilon \notin R^p, \quad \eta_n^\epsilon \notin R^q, \quad \eta_n^\epsilon \in H_{\rm FG}(\Phi) \ . \tag{50}$$

- ${\rm DSE}_q^\epsilon$ is a combinatorial Dyson–Schwinger equation defined by the family $\{B^+_{\eta_n^\epsilon,q}\}_{n\geq 1}$ and with the unique solution $X_\epsilon^q = \sum_{n\geq 0} X_n^{(\epsilon)q}$ for $\lambda g = 1$. There exists $N'_\epsilon \in \mathbb{N}$ such that for each $n \geq N'_\epsilon$, we have $d_{\rm cut}(X_n^{(j_2)}, X_n^{(\epsilon)q}) < \epsilon$. The triangle inequality of the cut distance metric leads us to

$$d_{\rm cut}(X_{{\rm DSE}_q^{(j_2)}}, X_\epsilon^q) < \epsilon \ . \tag{51}$$

Following Proof of Theorem 3.1, consider the lattices $(\mathcal{C}_p, \preccurlyeq)$ and $(\mathcal{C}_q, \preccurlyeq)$ of Hopf sub-subalgebras $H_{{\rm DSE}_p^{(j)}}$ and $H_{{\rm DSE}_q^{(l)}}$ associated with towers $\{{\rm DSE}_p^{(j)}\}_j$ and $\{{\rm DSE}_q^{(l)}\}_l$ of combinatorial Dyson–Schwinger equations generated by the initial equations ${\rm DSE}_p$ and ${\rm DSE}_q$, respectively. Following Corollary 3.3, we have cut-distance topological regions $\overline{R^p}$ and $\overline{R^q}$. They contain

Feynman diagrams which contribute to solutions of the equations in the towers $\{\mathrm{DSE}_p^{(j)}\}_j$ and $\{\mathrm{DSE}_q^{(l)}\}_l$. There exist some j, l such that equations $\mathrm{DSE}_p^{(j)}$ and $\mathrm{DSE}_q^{(l)}$ contribute to the relation (47). Set

(52)
$$j^* := \mathrm{Min}\{j : \ d_{\mathrm{cut}}(X_{\mathrm{DSE}_p^{(j)}}, Y) > 0 \,, \forall Y \in \overline{R^q}\} \,, \ l^* := \mathrm{Min}\{l : \ d_{\mathrm{cut}}(X, X_{\mathrm{DSE}_q^{(l)}}) > 0 \,, \ \forall X \in \overline{R^p}\}$$

to define the following (sub-)lattices.

- The sub-lattice $(\mathcal{C}_p^{j^*}, \preccurlyeq)$ is generated by the first j^* columns of the original lattice $(\mathcal{C}_p, \preccurlyeq)$.
- The sub-lattice $(\mathcal{C}_q^{l^*}, \preccurlyeq)$ is generated by the first l^* columns of the original lattice $(\mathcal{C}_q, \preccurlyeq)$.
- The lattice $(\mathcal{C}_{c_{pq}}^{j^*l^*}, \preccurlyeq)$ is generated by (i) Hopf (sub-)subalgebras in $(\mathcal{C}_p^{j^*}, \preccurlyeq)$ and $(\mathcal{C}_q^{l^*}, \preccurlyeq)$, (ii) Hopf (sub-)subalgebras corresponding to equations of types $\mathrm{DSE}_p^{\epsilon}$, $\mathrm{DSE}_q^{\epsilon}$ and (iii) Hopf (sub-)subalgebras corresponding to equations of type $\mathrm{DSE}_c^{(k)}$ with respect to a virtual particle c in the vacuum at the intermediate qft-states γ^c as a linear combination of the basic qft-states γ^p, γ^q. Chains

$$H_{\mathrm{DSE}_p} \preceq H_{\mathrm{DSE}_p^{(1)}} \preceq ... \preceq H_{\mathrm{DSE}_p^{(j^*)}} \ , \tag{53}$$

$$H_{\mathrm{DSE}_q} \preceq H_{\mathrm{DSE}_q^{(1)}} \preceq ... \preceq H_{\mathrm{DSE}_q^{(l^*)}} \tag{54}$$

in $(\mathcal{C}_p^{j^*}, \preccurlyeq)$ and $(\mathcal{C}_q^{l^*}, \preccurlyeq)$ are coupled in $(\mathcal{C}_{c_{pq}}^{j^*l^*}, \preccurlyeq)$ by one of the following sequences of Hopf algebra homomorphisms

$$H_{\mathrm{DSE}_p^{(j^*)}} \longrightarrow H_{\mathrm{DSE}_p^{\epsilon}} \longrightarrow H_{\mathrm{DSE}_c^{(k)}} \longrightarrow H_{\mathrm{DSE}_q^{\epsilon}} \longrightarrow H_{\mathrm{DSE}_q^{(l^*)}} \tag{55}$$

or

$$H_{\mathrm{DSE}_q^{(l^*)}} \longrightarrow H_{\mathrm{DSE}_q^{\epsilon}} \longrightarrow H_{\mathrm{DSE}_c^{(k)}} \longrightarrow H_{\mathrm{DSE}_p^{\epsilon}} \longrightarrow H_{\mathrm{DSE}_p^{(j^*)}} \ . \tag{56}$$

□ □

Remark 3.5. Following notations of Theorem 3.4 and its Proof, suppose DSE'_p and DSE'_q are another combinatorial Dyson–Schwinger equations such that they have γ^p and γ^q in their structures, respectively, and they generate the lattice $(\mathcal{C}_{c_{pq}}^{',j^*l^*}, \preccurlyeq)$.

- If DSE_p and DSE'_p are weakly isomorphic, DSE_q and DSE'_q are weakly isomorphic, then the lattices $(\mathcal{C}_{c_{pq}}^{j^*l^*}, \preccurlyeq)$ and $(\mathcal{C}_{c_{pq}}^{',j^*l^*}, \preccurlyeq)$ are isomorphic.
- If $d_{\mathrm{cut}}(X_{\mathrm{DSE}_p}, X_{\mathrm{DSE}'_p}) \neq 0$ or $d_{\mathrm{cut}}(X_{\mathrm{DSE}_q}, X_{\mathrm{DSE}'_q}) \neq 0$, then lattices $(\mathcal{C}_{c_{pq}}^{j^*l^*}, \preccurlyeq)$ and $(\mathcal{C}_{c_{pq}}^{',j^*l^*}, \preccurlyeq)$ determine different information transition packages.

Corollary 3.6. *Following notations in Theorems 3.1, 3.4 and Corollary 3.3, set $R^{c_{pq}}$ as the collection of Feynman diagrams in Φ which contribute to solutions of combinatorial Dyson–Schwinger equations of the types $\mathrm{DSE}_p^{\epsilon}$*

and $\mathrm{DSE}_q^{\epsilon}$ together with their related towers $\{\mathrm{DSE}_p^{\epsilon,(j)}\}_j$ and $\{\mathrm{DSE}_q^{\epsilon,(l)}\}_l$. Its completion with respect to the cut-distance topology is presented by $\overline{R^{c_{pq}}}$.

- *$\overline{R^{c_{pq}}}$ involves Feynman diagrams which contribute to solutions of combinatorial Dyson–Schwinger equations of the type $\mathrm{DSE}_c^{(k)}$ corresponding to virtual particles c in the vacuum at intermediate qft-states originated from the basic qft-states γ^p and γ^q.*
- *Thanks to Definition 2.3, Theorems 2.11 and 2.12 and Corollary 2.13, the lattice $(\mathcal{C}_{c_{pq}}^{j^*l^*}, \preccurlyeq)$ encodes quantum entanglement between all (virtual) particles which contribute to the region $\overline{R^{c_{pq}}}$ at states generated by the basic qft-states γ^p, γ^q and γ^c.*
- *Thanks to Theorems 2.11, 2.12 and Corollary 2.13, the cut-distance topological region $\overline{R^p} \cup \overline{R^q} \cup \overline{R^{c_{pq}}}$ determines separated but correlated space-time regions U_p, which has p at the qft-state γ^p and U_q, which has q at the qft-state γ^q.*

4. Intermediate algorithms associated with quantum entanglement

On the one hand, intermediate structures between programs and computable functions are studied in terms of Galois theory [41, 42] where intermediate algorithmic structures are concerned on the basis of automorphisms of programs. Galois theory is useful to explain the way that a subobject sits inside an object. The fundamental theorem of Galois theory of algorithms allows us to formulate a bijective correspondence between subgroups of the group of all automorphisms and intermediate algorithmic structures. On the other hand, combinatorial Dyson–Schwinger equations are important source of graded Hopf subalgebrs in gauge field theories. In the dual setting, it is possible to generate Lie subgroups from these Hopf subalgebras. The Hopf subalgebra H_{DSE} of an equation DSE is a Hopf ideal in the renormalization Hopf algebra. The dual of the quotient Hopf algebra $H_{\mathrm{FG}}(\Phi)/H_{\mathrm{DSE}}$ determines a Lie subgroup of the Lie group $\mathbb{G}_{\mathrm{FG}}(\mathbb{C})$ of diffeographisms of the physical theory. Thanks to the Manin approach to Halting problem via the BPHZ perturbative renormalization [20, 21], a theory of computation for non-perturbative parameters is introduced and developed. [25, 26, 27, 32]

In this section, we are going to lift lattices of Hopf sub-subalgebras given by Theorems 3.1 and 3.4 onto Lie groups. This new setting is useful to study quantum entanglement in gauge field theories in terms of flowcharts and intermediate algorithms in theory of computation.

Corollary 4.1. *There exists a lattice of Lie subgroups which encodes the quantum entanglement between a particle p at a basic qft-state γ^p and virtual particles at intermediate qft-states originated from γ^p in an interacting gauge field theory Φ.*

Proof. Following notations in Proof of Theorem 3.1, consider the lattice $(\mathcal{C}_p, \preceq)$ corresponding to the combinatorial Dyson–Schwinger equation DSE_p.

For each arbitrary pair of objects $H_{\mathrm{DSE}_p^{(k)}} \preceq H_{\mathrm{DSE}_p^{(l)}}$, there exists the natural injective Hopf algebra homomorphism $i_{kl} : H_{\mathrm{DSE}_p^{(k)}} \longrightarrow H_{\mathrm{DSE}_p^{(l)}}$. It determines the surjective morphism $\widetilde{i}_{kl} : \mathrm{Spec}(H_{\mathrm{DSE}_p^{(l)}}) \longrightarrow \mathrm{Spec}(H_{\mathrm{DSE}_p^{(k)}})$ of affine group schemes with respect to the contravariant functor Spec.

For each $H_{\mathrm{DSE}_p^{(k)}}$, the affine group scheme $\mathrm{Spec}(H_{\mathrm{DSE}_p^{(k)}})$ is a representable covariant functor from the category of commutative algebras to the category of groups. Set

$$G_{\mathrm{DSE}_p^{(k)}} = \mathrm{Spec}(H_{\mathrm{DSE}_p^{(k)}})(\mathbb{C}) = \mathrm{Hom}(H_{\mathrm{DSE}_p^{(k)}}, \mathbb{C}) \tag{57}$$

as the complex Lie group of characters corresponding to $H_{\mathrm{DSE}_p^{(k)}}$.

Follow the lattice (43) to define a new lattice $(\mathcal{G}_p, \succeq)$ of Lie subgroups in terms of the binary relation

$$G_s \succeq G_t \;\Leftrightarrow\; \exists\, G_1, ..., G_r \in \mathcal{G}_p \tag{58}$$

such that there exist surjective group homomorphisms

$$G_s \to G_1 \to G_2 \to ... \to G_r \to G_t \tag{59}$$

which connect G_s to G_t. For each $n \geq 1$, consider $G(X_1^j, ..., X_n^j)$ as the Lie subgroup corresponding to the free commutative graded Hopf sub-subalgebra $H(X_1^j, ..., X_n^j)$ of $H_{\mathrm{DSE}_p^{(j)}}$. The lattice $(\mathcal{G}_p, \succeq)$ is presented by

(60)

$$\begin{tikzcd}
... \arrow[r] & G(X_1^{j+1}) \arrow[r] & G(X_1^j) \arrow[r] & ... \arrow[r] & G(X_1^2) \arrow[r] & G(X_1^1) \arrow[r] & G(X_1) \\
... \arrow[u] \arrow[r] & G(X_1^{j+1}, X_2^{j+1}) \arrow[u] \arrow[r] & G(X_1^j, X_2^j) \arrow[u] \arrow[r] & ... \arrow[u] \arrow[r] & G(X_1^2, X_2^2) \arrow[u] \arrow[r] & G(X_1^1, X_2^1) \arrow[u] \arrow[r] & G(X_1, X_2) \arrow[u] \\
... \arrow[u] \arrow[r] & ... \arrow[u] \arrow[r] & ... \arrow[u] \arrow[r] & ... \arrow[u] \arrow[r] & ... \arrow[u] \arrow[r] & ... \arrow[u] \arrow[r] & ... \arrow[u] \\
... \arrow[u] \arrow[r] & G(X_1^{j+1}, ..., X_n^{j+1}) \arrow[u] \arrow[r] & G(X_1^j, ..., X_n^j) \arrow[u] \arrow[r] & ... \arrow[u] \arrow[r] & G(X_1^2, ..., X_n^2) \arrow[u] \arrow[r] & G(X_1^1, ..., X_n^1) \arrow[u] \arrow[r] & G(X_1, ..., X_n) \arrow[u] \\
... \arrow[u] \arrow[r] & ... \arrow[u] \arrow[r] & ... \arrow[u] \arrow[r] & ... \arrow[u] \arrow[r] & ... \arrow[u] \arrow[r] & ... \arrow[u] \arrow[r] & ... \arrow[u] \\
... \arrow[u] \arrow[r] & G_{\mathrm{DSE}_p^{(j+1)}} \arrow[u] \arrow[r] & G_{\mathrm{DSE}_p^{(j)}} \arrow[u] \arrow[r] & ... \arrow[u] \arrow[r] & G_{\mathrm{DSE}_p^{(2)}} \arrow[u] \arrow[r] & G_{\mathrm{DSE}_p^{(1)}} \arrow[u] \arrow[r] & G_{\mathrm{DSE}_p} \arrow[u]
\end{tikzcd}$$

$\square$ $\square$

Corollary 4.2. *There exists a lattice of Lie subgroups which encodes the quantum entanglement between space-time separated particles p at a basic qft-state γ^p and q at a basic qft-state γ^q in a (strongly coupled) interacting gauge field theory Φ.*

Proof. It is a direct result of Theorem 3.4 and Corollary 4.1. Following notations in Proof of Theorem 3.4, consider the lattice $(\mathcal{C}_{c_{pq}}^{j^* l^*}, \preceq)$. Lift the increasing chains (53), (54) onto groups to obtain the following decreasing chains of Lie groups.

$$G_{\mathrm{DSE}_p^{(j^*)}} \geq G_{\mathrm{DSE}_p^{(j^*-1)}} \geq ... \geq G_{\mathrm{DSE}_p^{(1)}} \geq G_{\mathrm{DSE}_p} \ , \tag{61}$$

$$G_{\mathrm{DSE}_q^{(l^*)}} \geq G_{\mathrm{DSE}_q^{(l^*-1)}} \geq ... \geq G_{\mathrm{DSE}_q^{(1)}} \geq G_{\mathrm{DSE}_q} \ . \tag{62}$$

These chains can be coupled to each other by using one of the following group homomorphisms

$$G_{\mathrm{DSE}_q^{(l^*)}} \longrightarrow G_{\mathrm{DSE}_q^{\epsilon}} \longrightarrow G_{\mathrm{DSE}_c^{(k)}} \longrightarrow G_{\mathrm{DSE}_p^{\epsilon}} \longrightarrow G_{\mathrm{DSE}_p^{(j^*)}} \tag{63}$$

or

$$G_{\mathrm{DSE}_p^{(j^*)}} \longrightarrow G_{\mathrm{DSE}_p^{\epsilon}} \longrightarrow G_{\mathrm{DSE}_c^{(k)}} \longrightarrow G_{\mathrm{DSE}_q^{\epsilon}} \longrightarrow G_{\mathrm{DSE}_q^{(l^*)}} \ . \tag{64}$$

$G_{\mathrm{DSE}_c^{(k)}}$ is the complex Lie group corresponding to $H_{\mathrm{DSE}_c^{(k)}}$ of the equation $\mathrm{DSE}_c^{(k)}$ with respect to a virtual particle c in the vacuum at an intermediate qft-state as a linear combination of basic qft-states γ^p, γ^q. Now build a new lattice $(\mathcal{G}_{c_{pq}}^{j^*l^*}, \succeq)$ of Lie subgroups generated by objects of the lattices $(\mathcal{G}_p^{j^*}, \succeq)$, $(\mathcal{G}_q^{l^*}, \succeq)$ together with Lie subgroups corresponding to Hopf subalgebras of types $H_{\mathrm{DSE}_p^{\epsilon}}$, $H_{\mathrm{DSE}_q^{\epsilon}}$ and $H_{\mathrm{DSE}_c^{(k)}}$. □ □

5. A renormalization group setting for quantum entanglement

The Connes–Kreimer Hopf algebraic renormalization has been lifted onto a universal categorical setting by Connes and Marcolli where they have encoded perturbative renormalization of gauge field theories in the language of differential Galois theory. They built a universal category of flat equisingular vector bundles which can encode counterterms and Renormalization Groups of physical theories [9]. This categorical setting has been also developed for the study of quantum motions to formulate non-perturbative counterterms in terms of systems of differential equations together with singularities [32].

In this section, we show that the Connes–Marcolli category is rich enough to encode quantum entanglement in interacting gauge field theories.

Lemma 5.1. *There exists a lattice of neutral Tannakian subcategories which encodes the quantum entanglement between a particle p at a basic qft-state γ^p and virtual particles at intermediate qft-states originated from γ^p in an interacting gauge field theory Φ.*

Proof. Following notations in Proofs of Theorem 3.1 and Corollary 4.1, consider the lattices $(\mathcal{C}_p, \preceq)$ and $(\mathcal{G}_p, \succeq)$ corresponding to the combinatorial Dyson–Schwinger equation DSE_p. For each arbitrary pair of objects $H_{\mathrm{DSE}_p^{(k)}} \preceq H_{\mathrm{DSE}_p^{(l)}}$, there exists the natural injective Hopf algebra homomorphism $i_{kl} : H_{\mathrm{DSE}_p^{(k)}} \longrightarrow H_{\mathrm{DSE}_p^{(l)}}$. It can be lifted onto the surjective group homomorphism $\bar{i}_{kl} : G_{\mathrm{DSE}_p^{(l)}} \longrightarrow G_{\mathrm{DSE}_p^{(k)}}$.

For each $G_{\mathrm{DSE}_p^{(l)}} \in (\mathcal{G}_p, \succeq)$, set $G^*_{\mathrm{DSE}_p^{(l)}} := G_{\mathrm{DSE}_p^{(l)}} \rtimes \mathbb{G}_m$ such that $\mathbb{G}_m$ is the multiplicative group which acts on the original group. Consider the category $\mathrm{Rep}_{G^*_{\mathrm{DSE}_p^{(l)}}}$ of finite dimensional representations of the complex Lie group $G^*_{\mathrm{DSE}_p^{(l)}}$ which is a neutral Tannakian category. The surjective morphism $\bar{i}_{kl}$ transforms the representation $\sigma : G_{\mathrm{DSE}_p^{(k)}} \longrightarrow GL_V$ to the representation $\sigma \circ \bar{i}_{kl} : G_{\mathrm{DSE}_p^{(l)}} \longrightarrow GL_V$. Therefore we obtain an exact fully faithful functor

$$\mathrm{Rep}_{G^*_{\mathrm{DSE}_p^{(k)}}} \longrightarrow \mathrm{Rep}_{G^*_{\mathrm{DSE}_p^{(l)}}} . \tag{65}$$

Now thanks to the lattice (60), we build a new lattice $(\mathrm{Cat}_p, \succeq)$ of neutral Tannakian subcategories in terms of the binary relation

$$\mathrm{Rep}_{H^*} \succeq \mathrm{Rep}_{K^*} \;\Leftrightarrow\; \exists\, \mathrm{Rep}_{H_1^*}, ..., \mathrm{Rep}_{H_t^*} \in \mathrm{Cat}_p \tag{66}$$

such that there exist exact fully faithful functors

$$\mathrm{Rep}_{K^*} \to \mathrm{Rep}_{H_1^*} \to ... \to \mathrm{Rep}_{H_t^*} \to \mathrm{Rep}_{H^*} \tag{67}$$

, determined by epimorphisms $\bar{i}_{kl}$, which connects Rep_{K^*} to Rep_{H^*}. □ □

Lemma 5.2. *There exists a lattice of neutral Tannakian subcategories which encodes the quantum entanglement between space-time separated particles p at a basic qft-state γ^p and q at a basic qft-state γ^q in a (strongly coupled) interacting gauge field theory Φ.*

Proof. Following notations in Proofs of Theorem 3.4, Corollary 4.2 and Lemma 5.1, consider lattices $(\mathcal{C}^{j^*l^*}_{c_{pq}}, \preceq)$ and $(\mathcal{G}^{j^*l^*}_{c_{pq}}, \succeq)$. Lift decreasing chains (61) and (62) onto the following chains of categories and exact fully faithful functors.

$$\mathrm{Rep}_{G^*_{\mathrm{DSE_p}}} \leq \mathrm{Rep}_{G^*_{\mathrm{DSE_p^{(1)}}}} \leq ... \leq \mathrm{Rep}_{G^*_{\mathrm{DSE_p^{(j^*-1)}}}} \leq \mathrm{Rep}_{G^*_{\mathrm{DSE_p^{(j^*)}}}} , \tag{68}$$

$$\mathrm{Rep}_{G^*_{\mathrm{DSE_q}}} \leq \mathrm{Rep}_{G^*_{\mathrm{DSE_q^{(1)}}}} \leq ... \leq \mathrm{Rep}_{G^*_{\mathrm{DSE_q^{(l^*-1)}}}} \leq \mathrm{Rep}_{G^*_{\mathrm{DSE_q^{(l^*)}}}} . \tag{69}$$

These two chains can be coupled to each other by using one the following sequences of exact fully faithful functors.

$$\mathrm{Rep}_{G^*_{\mathrm{DSE}_q^{(l^*)}}} \longrightarrow \mathrm{Rep}_{G^*_{\mathrm{DSE}_q^{\epsilon}}} \longrightarrow \mathrm{Rep}_{G^*_{\mathrm{DSE}_c^{(k)}}} \longrightarrow \mathrm{Rep}_{G^*_{\mathrm{DSE}_p^{\epsilon}}} \longrightarrow \mathrm{Rep}_{G^*_{\mathrm{DSE}_p^{(j^*)}}} \tag{70}$$

or

$$\mathrm{Rep}_{G^*_{\mathrm{DSE}_p^{(j^*)}}} \longrightarrow \mathrm{Rep}_{G^*_{\mathrm{DSE}_p^{\epsilon}}} \longrightarrow \mathrm{Rep}_{G^*_{\mathrm{DSE}_c^{(k)}}} \longrightarrow \mathrm{Rep}_{G^*_{\mathrm{DSE}_q^{\epsilon}}} \longrightarrow \mathrm{Rep}_{G^*_{\mathrm{DSE}_q^{(l^*)}}} . \tag{71}$$

This information is enough to build a new lattice $(\mathrm{Cat}^{j^*l^*}_{c_{pq}}, \succeq)$ of neutral Tannakian subcategories which contains $(\mathrm{Cat}_p, \succeq)$ and $(\mathrm{Cat}_q, \succeq)$ as sublattices. □ □

Theorem 5.3. *The Connes–Marcolli category encodes quantum entanglement in (strongly coupled) gauge field theories.*

Proof. The universal Connes–Marcolli category is a neutral Tannakian category which is isomorphic to the category $\mathrm{Rep}_{\mathbb{U}^*}$ of all finite dimensional representations of the universal affine group scheme $\mathbb{U}^*$. The covariant functor $\mathbb{U}^*$ is representable by the universal Hopf algebra of renormalization $H_{\mathbb{U}}$. This particular Hopf algebra is isomorphic to the linear space of noncommutative polynomials in variables $f_n, n \in \mathbb{N}$ equipped with the shuffle product. [9]

For a gauge field theory Φ with the renormalization Hopf algebra $H_{\mathrm{FG}}(\Phi)$, consider the affine group scheme $\mathbb{G}_\Phi$ which is representable by $H_{\mathrm{FG}}(\Phi)$. The category $\mathrm{Rep}_{\mathbb{G}^*_\Phi}$ can be embedded in $\mathrm{Rep}_{\mathbb{U}^*}$ as a subcategory [9]. In addition, for any combinatorial Dyson–Schwinger equation DSE in Φ with the corresponding Hopf subalgebra H_{DSE}, consider the affine group scheme $\mathbb{G}_{\mathrm{DSE}}$ which is representable by H_{DSE}. The category $\mathrm{Rep}_{\mathbb{G}^*_{\mathrm{DSE}}}$ can be embedded in $\mathrm{Rep}_{\mathbb{U}^*}$ as a subcategory [32]. Therefore the category $\mathrm{Rep}_{\mathbb{U}^*}$ encodes physical information of perturbation and non-perturbation parts of the physical theory Φ.

Thanks to Lemmas 5.1 and 5.2, the chains (70) and (71) of subcategories encapsulate quantum entanglement via one of the relations $\pi_{l^*} = r^{pq}_{j^*l^*} \circ \pi_{j^*}$ or $\pi_{j^*} = r^{qp}_{l^*j^*} \circ \pi_{l^*}$ such that

$$\pi_{j^*} : \mathrm{Rep}_{\mathbb{U}^*} \to \mathrm{Rep}_{G^*_{\mathrm{DSE}_p^{(j^*)}}}, \quad \pi_{l^*} : \mathrm{Rep}_{\mathbb{U}^*} \to \mathrm{Rep}_{G^*_{\mathrm{DSE}_q^{(l^*)}}},$$

$$r^{pq}_{j^*l^*} : \mathrm{Rep}_{G^*_{\mathrm{DSE}_p^{(j^*)}}} \to \mathrm{Rep}_{G^*_{\mathrm{DSE}_q^{(l^*)}}}, \quad r^{qp}_{l^*j^*} : \mathrm{Rep}_{G^*_{\mathrm{DSE}_q^{(l^*)}}} \to \mathrm{Rep}_{G^*_{\mathrm{DSE}_p^{(j^*)}}}. \tag{72}$$

$\mathrm{Rep}_{G^*_{\mathrm{DSE}_p^{(j^*)}}}$ and $\mathrm{Rep}_{G^*_{\mathrm{DSE}_q^{(l^*)}}}$ can be embedded in $\mathrm{Rep}_{\mathbb{U}^*}$ as subcategories. □ □

Corollary 5.4. *There exists a collection of mixed Tate motives which encodes quantum entanglement in a gauge field theory.*

Proof. $\mathrm{Rep}_{\mathbb{U}^*}$ is equivalent to the category $\mathcal{TM}_{\mathrm{mix}}(\mathrm{Spec}\ \mathcal{O}[1/N])$ of mixed Tate motives (i.e. Proposition 1.110, Corollary 1.111 in [9]). Therefore, thanks to Theorem 5.3, $\mathrm{Rep}_{G^*_{\mathrm{DSE}_p^{(j^*)}}}$ and $\mathrm{Rep}_{G^*_{\mathrm{DSE}_q^{(l^*)}}}$ determine subcategories of the category $\mathcal{TM}_{\mathrm{mix}}(\mathrm{Spec}\ \mathcal{O}[1/N])$. These subcategories contain mixed Tate motives associated with the complex Lie groups $G_{\mathrm{DSE}_p^{(j^*)}}$ and $G_{\mathrm{DSE}_q^{(l^*)}}$ corresponding to the equations $\mathrm{DSE}_p^{(j^*)}$ and $\mathrm{DSE}_q^{(l^*)}$. □ □

Thanks to Theorem 5.3, quantum entanglement in a gauge field theory can be studied under a renormalization group. This renormalization group is useful to show the invariance of our lattice model under changing the scales of the energy.

Corollary 5.5. *Quantum entanglement between space-time separated particles at different (intermediate or general) qft-states is invariant under changing the energy scale in a gauge field theory.*

Proof. Following notations in Proof of Theorem 5.3, the universal Hopf algebra of renormalization $H_{\mathbb{U}}$ is the graded dual of the universal enveloping algebra of the free graded Lie algebra $L_{\mathbb{U}}$ which is generated by elements e_{-n} of degree $-n$ for each $n > 0$. The corresponding pro-unipotent affine group scheme $\mathbb{U}$ is determined by Milnor–Moore Theorem. The sum $e := \sum_n e_{-n}$ is an element of $L_{\mathbb{U}}$ which can be lifted onto the morphism $\mathrm{rg} : \mathbb{G}_a \to \mathbb{U}$. [9]

Following notations in Proofs of Theorems 3.1, 3.4 and Corollaries 4.1, 4.2, consider a basic combinatorial Dyson–Schwinger equation DSE_p with the corresponding complex Lie group G_{DSE_p}. Apply Theorem 1.106 in [9] to determine a graded representation $\tau : \mathbb{U}^*(\mathbb{C}) \to G^*_{\mathrm{DSE}_p}$. We apply the map $\tau \circ \mathrm{rg}$ to build a renormalization group $\{F_t^{\mathrm{DSE}_p}\}_{t\in\mathbb{R}}$ with respect to the equation DSE_p. It encodes the behavior of DSE_p under changing the energy scale during the renormalization of its solution. This framework associates a renormalization group $\{F_t^{\mathrm{DSE}_p^{(j)}}\}_{t\in\mathbb{R}}$ to each equation $\mathrm{DSE}_p^{(j)}$, $j \geq 1$ in the tower $\{\mathrm{DSE}_p^{(j)}\}_{j\geq 1}$ built by the relation (39). Follow the same process to build a tower of coupled renormalization groups with respect to the lattice $(\mathcal{G}^{j^*l^*}_{c_{pq}}, \succeq)$ given by Corollaries 4.1 and 4.2. Define a new lattice $(\mathbf{RG}_{c_{pq}}, \succeq)$ of renormalization groups corresponding to those combinatorial Dyson–Schwinger equations which contribute to the region $\overline{R^p} \cup \overline{R^q} \cup \overline{R^{c_{pq}}}$. □ □

6. Conclusion

In this research we applied the cut-distance topologically enriched renormalization Hopf algebra $H^{\mathrm{cut}}_{\mathrm{FG}}(\Phi)$ and Feynman graphon representations of solutions of quantum motions to formulate a new lattice setting for the description of quantum entanglement in a gauge field theory Φ as a fundamental property of cut-distance topological regions of Feynman diagrams, as space-time graphs, around particles. (See Corollaries 2.13, 3.3 and 3.6). Hopf subalgebras of combinatorial Dyson–Schwinger equations, as particular subspaces of the renormalization Hopf algebra, are associated to space-time regions. See Theorems 2.11, 2.12, 3.1 and 3.4 and Corollary 2.13.

- Quantum entanglement is a fundamental concept in theory of quantum information where random processes are studied and developed [13, 18, 19]. Feynman graphon models led us to formulate random graph processes and homomorphism densities for the study of solutions of quantum motions [26, 28, 29]. Here we formulated the Fisher information metric on the finite dimensional subspaces of states in a gauge field theory in terms of homomorphism densities of Feynman graphons (i.e. Theorem 2.9). Then we developed our setting to associate a statistical manifold to the space of 1PI Green's functions of Φ. The Fisher information metrics (18) and (37) and lattices of

Hopf subalgebras, Lie subgroups and Tannakian subcategories are new tools to analyze quantum entanglement in terms of correlations between cut-distance topological regions of Feynman diagrams which contribute to solutions of towers of combinatorial Dyson–Schwinger equations. See Theorems 2.16, 3.1, 3.4 and 5.3 and Corollaries 4.1, 4.2. In addition, the computational complexities of non-perturbative solutions of combinatorial Dyson–Schwinger equations are considered where some new complexity parameters together with systems of graph languages are built and developed [25, 27]. They are useful tools to optimize the construction processes of infinite random graphs which encode solutions of quantum motions [25, 26]. Thanks to these background, it is now possible to associate some complexity parameters to correlations between separated regions in $H^{\mathrm{cut}}_{\mathrm{FG}}(\Phi)$. It leads us to recognize the optimal transitional processes between space-time separated particles at different qft-states.

- We built a new lattice of Lie subgroups which encodes quantum entanglement (i.e. Corollaries 4.1 and 4.2). It allows us to determine intermediate algorithms which contribute to the structure of correlations between separated regions of space-time.
- The Connes–Marcolli universal category of flat equi-singular vector bundles encodes all physical information of renormalizable physical theories and also, quantum motions via systems of differential equations [9, 32]. This category plays a fundamental role to relate quantum field theory to theory of motives. Theorem 5.3 and Corollaries 5.4 and 5.5 show that this category is rich enough to encode quantum entanglement in gauge filed theories under different energy scales.

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